Name: ______________________________

Math 314 Linear Algebra (Bueler) Monday, 28 March 2022

# Midterm Exam 2

**No book, notes, electronics, calculator, or internet access. 100 points possible. 65 minutes maximum.**

**1.** Here is a matrix and its row-reduced echelon form:

$$A = \begin{bmatrix} -2 & -4 & 1 & 2 & -1 \\ -1 & -2 & 2 & 1 & -1 \\ 0 & 0 & 3 & 0 & 3 \end{bmatrix} \qquad \to \qquad R = \begin{bmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**(a)** (*5 pts*) What is the **dimension** of the row space $C(A^\top)$? **Provide a basis** for $C(A^\top)$. (*Suggestion:* Write your basis as $C(A^\top) = \text{span}\{\dots\}$ with particular vectors.)

$\dim C(A^\top) =$ ☐

**(b)** (*5 pts*) What is the **dimension** of the column space $C(A)$? **Provide a basis**.

$\dim C(A) =$ ☐

**(c)** (*5 pts*) What is the **dimension** of the null space $N(A)$? **Provide a basis**.

$\dim N(A) =$ ☐

**2.** (*10 pts*) Suppose $A$ is an $m$ by $n$ matrix. Show that the null space $N(A)$ and the row space $C(A^\top)$ are orthogonal, as subspaces of the vector space $\mathbb{R}^n$. (*Hint.* What is a good way to write a generic vector from $C(A^\top)$?)

**3.** (*6 pts*) Suppose $A$ is an $m$ by $n$ matrix. Show that $A^\top A$ is symmetric.

**4.** Consider the overdetermined linear system "$A\mathbf{v} = \mathbf{b}$" with

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 10 \\ -1 \\ -6 \end{bmatrix}.$$

**(a)** (*10 pts*) Write down the normal equations for this system.

**(b)** (*6 pts*) The solution of the normal equations is $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Is $\mathbf{b}$ in the column space $C(A)$ for this system? How do you know?

**5.** (*10 pts*) Consider this linear system $A\mathbf{x} = \mathbf{b}$:

$$x_1 + 2x_2 + 2x_3 + x_4 = 9$$
$$3x_1 + 6x_2 + 4x_3 + x_4 = 17$$

Here is the row-reduced echelon form of the augmented matrix:

$$[A\ \ \mathbf{b}] \qquad \to \qquad [R\ \ \mathbf{d}] = \begin{bmatrix} 1 & 2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 5 \end{bmatrix}$$

What is the general solution of the system? Show your work.

**6.** (*6 pts*) Show that if $A$ is any matrix and $\mathbf{x}$ is in $N(A)$ then $\mathbf{x}$ is in $N(A^\top A)$.

**7.** *(8 pts)* $M_3$ is the vector space of all 3 by 3 matrices. Give a basis for the subspace $S$ of symmetric matrices.

**8.** *(6 pts)* Suppose $A$ is any $m$ by $n$ matrix with full rank. Let $P = A(A^\top A)^{-1}A^\top$, the projection onto $C(A)$. Show that $P^2 = P$.

**9.** *(8 pts)* Suppose $I$ is the 3 by 3 identity matrix and $O$ is the 2 by 3 zero matrix. Consider the matrix

$$A = \begin{bmatrix} I & I \\ O & O \end{bmatrix},$$

which is 5 by 6. What are the dimensions of the four subspaces?

$\dim C(A^\top) =$ [ ] $\dim C(A) =$ [ ]

$\dim N(A) =$ [ ] $\dim N(A^\top) =$ [ ]

**10.** Consider the line through the origin in $\mathbb{R}^2$ along the vector $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Suppose $\mathbf{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

**(a)** (*5 pts*) Sketch $\mathbf{a}$, the line along $\mathbf{a}$, and the vector $\mathbf{b}$, all on the same axes. (*Try to make your sketch to scale!*) Add $P\mathbf{b} = \hat{x}\mathbf{a}$, the projection of $\mathbf{b}$ onto the line.

**(b)** (*5 pts*) Compute $\hat{x}$ and $P\mathbf{b} = \hat{x}\mathbf{a}$.

**(c)** (*5 pts*) Compute the projection matrix $P$.

**Extra Credit.** (*3 pts*) The complete solution to $A\mathbf{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Find $A$. Show your work.

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