Name: SOLUTIONS

Math 314 Linear Algebra (Bueler) Monday, 28 March 2022

Midterm Exam 2

No book, notes, electronics, calculator, or internet access. 100 points possible. 65 minutes maximum.

1. Here is a matrix and its row-reduced echelon form:

$$A = \begin{bmatrix} -2 & -4 & 1 & 2 & -1 \\ -1 & -2 & 2 & 1 & -1 \\ 0 & 0 & 3 & 0 & 3 \end{bmatrix} \quad \to \quad R = \begin{bmatrix} 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (*5 pts*) What is the **dimension** of the row space $C(A^\top)$? **Provide a basis** for $C(A^\top)$. (*Suggestion:* Write your basis as $C(A^\top) = \text{span}\{\dots\}$ with particular vectors.)

$\dim C(A^\top) = \boxed{2}$

$$C(A^T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(b) (*5 pts*) What is the **dimension** of the column space $C(A)$? **Provide a basis**.

$\dim C(A) = \boxed{2}$

$$C(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$$

(c) (*5 pts*) What is the **dimension** of the null space $N(A)$? **Provide a basis**.

$\dim N(A) = \boxed{3}$

$$x_1 + 2x_2 - x_4 + x_5 = 0$$
$$x_3 + x_5 = 0$$

$$N(A) = \text{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

2. (*10 pts*) Suppose A is an m by n matrix. Show that the null space $N(A)$ and the row space $C(A^\top)$ are orthogonal, as subspaces of the vector space $\mathbb{R}^n$. (*Hint.* What is a good way to write a generic vector from $C(A^\top)$?)

Let $\vec{x}$ be in $N(A)$, so $A\vec{x} = \vec{0}$.

Let $\vec{z}$ be in $C(A^T)$ so $\vec{z} = A^T\vec{y}$ for some $\vec{y}$.

Then

$$\vec{x}^T\vec{z} = \vec{x}^T(A^T\vec{y}) = (\vec{x}^T A^T)\vec{y}$$

$$= (A\vec{x})^T\vec{y} = \vec{0}^T\vec{y} = 0$$

So $\vec{x}$, $\vec{z}$ are orthogonal. Thus $N(A)$ and $C(A^T)$ are orthogonal subspaces.

3. (*6 pts*) Suppose A is an m by n matrix. Show that $A^\top A$ is symmetric.

A matrix is symmetric if it equals its own transpose. But

$$(A^TA)^T = A^T(A^T)^T = A^TA$$

So A^TA is symmetric.

4. Consider the overdetermined linear system "$A\mathbf{v} = \mathbf{b}$" with

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 2 \\ -1 & 1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 10 \\ -1 \\ -6 \end{bmatrix}.$$

(a) (*10 pts*) Write down the normal equations for this system.

$$A^TA = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 10 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 1 & 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 10 \\ -1 \\ -6 \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

<u>normal equations:</u>

$$3v_1 + 2v_2 = 5$$

$$2v_1 + 10v_2 = 12$$

(b) (*6 pts*) The solution of the normal equations is $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Is $\mathbf{b}$ in the column space $C(A)$ for this system? How do you know?

Notice $A\vec{v} = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 0 \end{bmatrix} \neq \vec{b}$. So $A\vec{v} = P\vec{b}$ is not equal to $\vec{b}$. So $\vec{b}$ is <u>not</u> in $C(A)$.

5. (*10 pts*) Consider this linear system $A\mathbf{x} = \mathbf{b}$:

$$x_1 + 2x_2 + 2x_3 + x_4 = 9$$
$$3x_1 + 6x_2 + 4x_3 + x_4 = 17$$

Here is the row-reduced echelon form of the augmented matrix:

$$[A\ \ \mathbf{b}] \quad \to \quad [R\ \ \mathbf{d}] = \begin{bmatrix} 1 & 2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 5 \end{bmatrix}$$

What is the general solution of the system? Show your work.

$$\vec{x}_p = \begin{bmatrix} -1 \\ 0 \\ 5 \\ 0 \end{bmatrix}, \quad \vec{s}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{s}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

general solution:

$$\vec{x} = \vec{x}_p + c_1\vec{s}_1 + c_2\vec{s}_2 = \begin{bmatrix} -1 - 2c_1 + c_2 \\ c_1 \\ 5 - c_2 \\ c_2 \end{bmatrix}$$

for any c_1, c_2

6. (*6 pts*) Show that if A is any matrix and $\mathbf{x}$ is in $N(A)$ then $\mathbf{x}$ is in $N(A^\top A)$.

If $\vec{x}$ is in $N(A)$ then $A\vec{x} = \vec{0}$.

But then

$$(A^T A)\vec{x} = A^T(A\vec{x}) = A^T\vec{0} = \vec{0}$$

so $\vec{x}$ is also in $N(A^T A)$.

7. (*8 pts*) M_3 is the vector space of all 3 by 3 matrices. Give a basis for the subspace S of symmetric matrices.

$$S = \text{span}\left\{ \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}, \begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix} \right\}$$

(6 dimensions)

8. (*6 pts*) Suppose A is any m by n matrix with full rank. Let $P = A(A^\top A)^{-1}A^\top$, the projection onto $C(A)$. Show that $P^2 = P$.

$$P^2 = A(A^TA)^{-1}\underbrace{A^TA(A^TA)^{-1}}_{=I}A^T$$

$$= A(A^TA)^{-1}A^T = P$$

9. (*8 pts*) Suppose I is the 3 by 3 identity matrix and O is the 2 by 3 zero matrix. Consider the matrix

$$A = \begin{bmatrix} I & I \\ O & O \end{bmatrix},$$

A is 5×3

which is 5 by 6. What are the dimensions of the four subspaces?

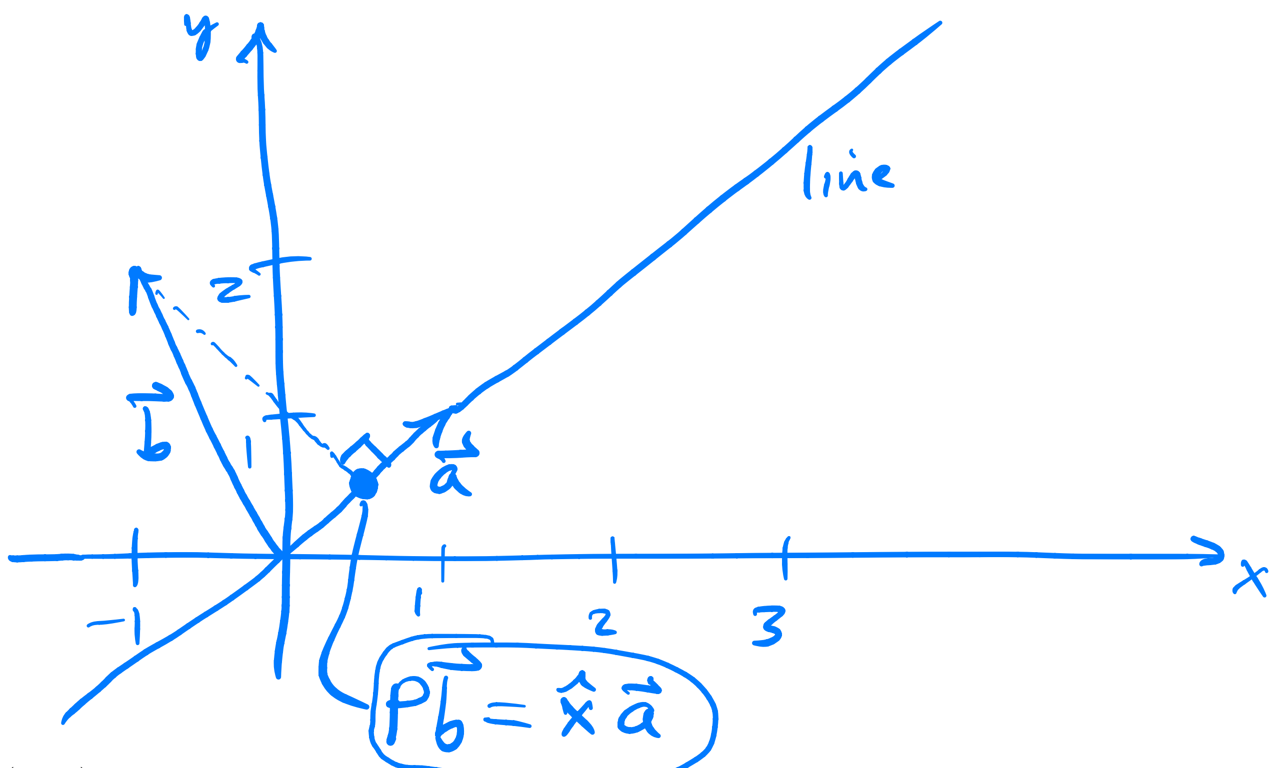
$r=3 \rightarrow$ $\dim C(A^\top) = \boxed{3}$ $\qquad \dim C(A) = \boxed{3}$

$\dim N(A) = \boxed{3}$ $\qquad \dim N(A^\top) = \boxed{2}$

↑ 3 free variables $\qquad$ ↑ $m-r = 5-3$

10. Consider the line through the origin in $\mathbb{R}^2$ along the vector $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Suppose $\mathbf{b} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

(a) (*5 pts*) Sketch $\mathbf{a}$, the line along $\mathbf{a}$, and the vector $\mathbf{b}$, all on the same axes. (*Try to make your sketch to scale!*) Add $P\mathbf{b} = \hat{x}\mathbf{a}$, the projection of $\mathbf{b}$ onto the line.



(b) (*5 pts*) Compute $\hat{x}$ and $P\mathbf{b} = \hat{x}\mathbf{a}$.

$$\hat{x} = \frac{\vec{a}^T\vec{b}}{\vec{a}^T\vec{a}} = \frac{[1\ \ 1]\begin{bmatrix} -1 \\ 2 \end{bmatrix}}{[1\ \ 1]\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \frac{-1+2}{2} = \frac{1}{2}$$

so $P\vec{b} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$

(c) (*5 pts*) Compute the projection matrix P.

$$P = \frac{\vec{a}\,\vec{a}^T}{\vec{a}^T\vec{a}} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

Extra Credit. (*3 pts*) The complete solution to $A\mathbf{x} = \begin{bmatrix}1\\3\end{bmatrix}$ is $\mathbf{x} = \begin{bmatrix}1\\0\end{bmatrix} + c\begin{bmatrix}0\\1\end{bmatrix}$. Find A. Show your work.

A is 2x2

$A\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$ so $A = \begin{bmatrix}1 & \alpha\\3 & \beta\end{bmatrix}$

but $A\begin{bmatrix}0\\1\end{bmatrix} = \vec{0}$ so $\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \vec{0}$

<u>so:</u>

$$A = \begin{bmatrix}1 & 0\\3 & 0\end{bmatrix}$$

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