Name: ______________________________

Math 314 Linear Algebra (Bueler) Monday, 14 February 2022

Midterm Exam 1

No book, notes, electronics, calculator, or internet access. 100 points possible. 65 minutes maximum.

1. Consider the following linear system $A\mathbf{x} = \mathbf{b}$:

$$\begin{aligned} 2x_1 + x_2 \qquad\quad &= 6 \\ -2x_1 + 3x_2 + 2x_3 &= 0 \\ 2x_1 + 9x_2 + 9x_3 &= 13 \end{aligned}$$

(a) (*10 pts*) Solve the linear system by *elimination* and then *back-substitution*. Use the standard algorithm. Show your work, and in particular show, as an intermediate stage, the triangular system which you get after elimination.

(b) (*4 pts*) From part **(a)**, what elimination matrix E_{32} does the row operation which generated a zero in the $(3, 2)$ location?

(c) (*4 pts*) From part **(a)**, what three numbers were the pivots?

(d) (*6 pts*) The computation in part **(a)** can regarded as factoring $A = LU$. What lower triangular matrix L and upper triangular matrix U were computed?

(e) (*2 pts*) Multiply LU and confirm you get the original matrix A.

2. (*10 pts*) Suppose A is an invertible $n \times n$ matrix which has a known LU factorization into a lower triangular matrix L and an upper triangular matrix U. That is, suppose $A = LU$. Explain the steps, and name the algorithms, which you would use to solve a linear system $A\mathbf{x} = \mathbf{b}$. (*Hint. The elimination stage has already been done. Don't propose to redo it!*)

3. (*8 pts*) I have claimed that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Show (confirm) that this formula is correct. (*Hint. A matrix multiplication suffices.*)

4. True or false? **Circle one.** Give a short justification if true, and a counterexample if false.

(a) (*3 pts*) A matrix with two equal columns is not invertible.

TRUE

FALSE

(b) (*3 pts*) Every triangular matrix with 1's down the main diagonal is invertible.

TRUE

FALSE

(c) (*3 pts*) If A is symmetric, so $a_{ij} = a_{ji}$, then A is invertible.

TRUE

FALSE

(d) (*3 pts*) If AB and BA are defined then A and B are square.

TRUE

FALSE

(e) (*3 pts*) If A and B are square matrices of the same size then $(A+B)^2 = A^2 + 2AB + B^2$.

TRUE

FALSE

5. **(a)** (*10 pts*) Invert this matrix by the Gauss-Jordan method:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Please show your work!

(b) (*2 pts*) What is the determinant of the matrix in part **(a)**?

6. (*6 pts*) Find the angle between these vectors. (*Hint. You can write the answer in terms of an inverse trigonometric function, but otherwise everything should be simplified.*)

$$\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

7. (*6 pts*) Sketch the column picture of this linear system. Noting that the solution values (x, y) play a particular role in this picture, find them and then show the solution in the sketch:

$$\begin{aligned} x - 2y &= 0 \\ x + y &= 3 \end{aligned}$$

(*Hint. Clearly sketch and label the 3 known vectors. Now, how to combine 2 of them to get the third?*)

8. Consider the linear system

$$\begin{aligned} ax - 2y &= 1 \\ x + 4y &= 3 \end{aligned}$$

(a) (*4 pts*) For which number a does elimination break down permanently, so that there are no solutions?

(b) (*4 pts*) For which number a does elimination break down temporarily, so that a row swap allows a solution?

(c) (*3 pts*) For the value of a found in part **(b)**, solve the system.

Extra Credit. (*3 pts*) Find the quadratic polynomial $p(x) = a + bx + cx^2$ which passes through the points $(-1, 1)$, $(1, 5)$, $(3, 17)$.

9. (*6 pts*) Suppose $L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$. What is L^{-1}?

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