Name: SOLUTIONS

Math 314 Linear Algebra (Bueler) Monday, 14 February 2022

Midterm Exam 1

CORRECTED

No book, notes, electronics, calculator, or internet access. 100 points possible. 65 minutes maximum.

1. Consider the following linear system $A\mathbf{x} = \mathbf{b}$:

$$\begin{aligned} 2x_1 + x_2 \qquad &= 6 \\ -2x_1 + 3x_2 + 2x_3 &= 0 \\ 2x_1 + 9x_2 + 9x_3 &= 13 \end{aligned}$$

$\Leftarrow A = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 3 & 2 \\ 2 & 9 & 9 \end{bmatrix}$

(a) (*10 pts*) Solve the linear system by *elimination* and then *back-substitution*. Use the standard algorithm. Show your work, and in particular show, as an intermediate stage, the triangular system which you get after elimination.

elimination

$$\begin{bmatrix} 2 & 1 & 0 & 6 \\ -2 & 3 & 2 & 0 \\ 2 & 9 & 9 & 13 \end{bmatrix}$$

$R_2 \Leftarrow R_2 + R_1$

$R_3 \Leftarrow R_3 - R_1$

$$\begin{bmatrix} 2 & 1 & 0 & 6 \\ 0 & 4 & 2 & 6 \\ 0 & 8 & 9 & 7 \end{bmatrix}$$

$\ell_{21} = -1$

$\ell_{31} = 1$

triangular system $U\vec{x} = \vec{z}$

$R_3 \Leftarrow R_3 - 2R_2$

$$\begin{bmatrix} 2 & 1 & 0 & 6 \\ 0 & 4 & 2 & 6 \\ 0 & 0 & 5 & -5 \end{bmatrix}$$

$\ell_{32} = 2$

back-subst.

$5x_3 = -5 \quad \therefore \; x_3 = -1$

$4x_2 + 2x_3 = 6 \quad \therefore \; x_2 = \dfrac{6-2(-1)}{4} = 2$

$2x_1 + x_2 \qquad = 6 \quad \therefore \; x_1 = \dfrac{6-(2)}{2} = 2$

$\therefore \; \vec{x} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$

(b) (*4 pts*) From part **(a)**, what elimination matrix E_{32} does the row operation which generated a zero in the $(3, 2)$ location?

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$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

(c) (*4 pts*) From part **(a)**, what three numbers were the pivots?

2, 4, 5

(d) (*6 pts*) The computation in part **(a)** can regarded as factoring $A = LU$. What lower triangular matrix L and upper triangular matrix U were computed?

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 4 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

(e) (*2 pts*) Multiply LU and confirm you get the original matrix A.

$$LU = \begin{bmatrix} 2 & 1 & 0 \\ -2 & 3 & 2 \\ 2 & 9 & 9 \end{bmatrix} = A \quad \checkmark$$

2. (*10 pts*) Suppose A is an invertible $n \times n$ matrix which has a known LU factorization into a lower triangular matrix L and an upper triangular matrix U. That is, suppose $A = LU$. Explain the steps, and name the algorithms, which you would use to solve a linear system $A\mathbf{x} = \mathbf{b}$. (*Hint. The elimination stage has already been done. Don't propose to redo it!*)

[already done: ① $A = LU$ so $LU\vec{x} = \vec{b}$]

② solve $L\vec{c} = \vec{b}$ by forward substitution

③ solve $U\vec{x} = \vec{c}$ by back substitution

3. (*8 pts*) I have claimed that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}, \quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Show (confirm) that this formula is correct. (*Hint. A matrix multiplication suffices.*)

$$A^{-1}A = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \frac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & -bc+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \checkmark$$

4. True or false? **Circle one.** Give a short justification if true, and a counterexample if false.

(a) (*3 pts*) A matrix with two equal columns is not invertible.

TRUE

FALSE

An invertible matrix must have linearly-independent columns.

(b) (*3 pts*) Every triangular matrix with 1's down the main diagonal is invertible.

TRUE

FALSE

Since all pivots are nonzero, we can invert the matrix.

(c) (*3 pts*) If A is symmetric, so $a_{ij} = a_{ji}$, then A is invertible.

TRUE

FALSE

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not invertible, but it is symmetric

[the zero matrix works here too]

(d) (*3 pts*) If AB and BA are defined then A and B are square.

TRUE

FALSE

$A = \begin{bmatrix} 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are not square,

but $AB = \begin{bmatrix} 5 \end{bmatrix}$ and $BA = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ are defined

(e) (*3 pts*) If A and B are square matrices of the same size then $(A + B)^2 = A^2 + 2AB + B^2$.

TRUE

FALSE

$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ show $AB \neq BA$,

so $(A+B)^2 - (A^2 + 2AB + B^2) = A^2 + AB + BA + B^2 - A^2 - 2AB - B^2$

$= BA - AB \neq 0.$

5. **(a)** (*10 pts*) Invert this matrix by the Gauss-Jordan method:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Please show your work!

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 5 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The Pivots

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 & -5 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

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$$\therefore \quad A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

(check: $AA^{-1} = I$ ✓)

(b) (*2 pts*) What is the determinant of the matrix in part **(a)**?

$$\det(A) = 1 \cdot 1 \cdot 1 = 1$$

(product of pivots)

6. (*6 pts*) Find the angle between these vectors. (*Hint. You can write the answer in terms of an inverse trigonometric function, but otherwise everything should be simplified.*)

$$\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|} = \frac{2-1-2}{\sqrt{9}\sqrt{3}} = \frac{-1}{3\sqrt{3}}$$

$$\theta = \arccos\left(\frac{-1}{3\sqrt{3}}\right)$$

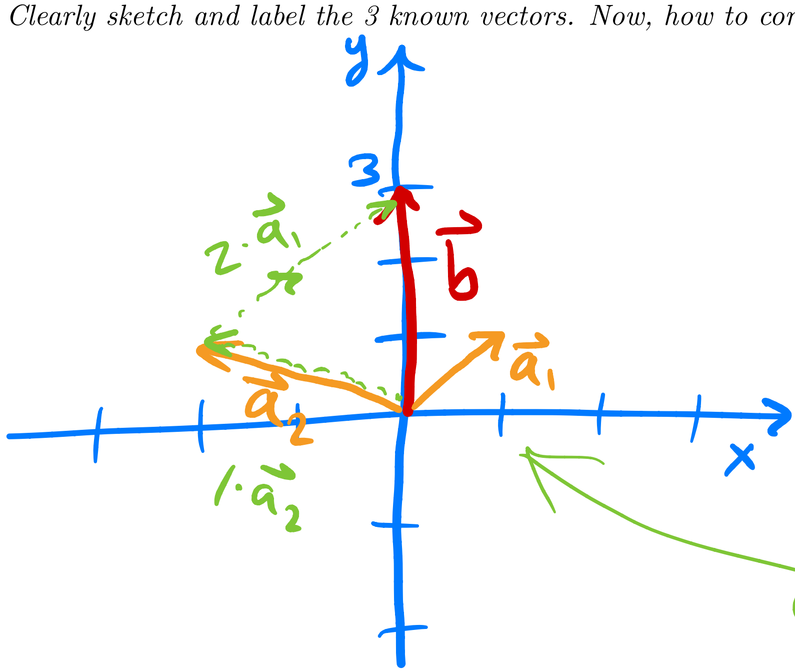
note $\frac{\pi}{2} < \theta < \pi$

since $\cos\theta < 0$

7. (*6 pts*) Sketch the column picture of this linear system. Noting that the solution values (x, y) play a particular role in this picture, find them and then show the solution in the sketch:

$$\begin{aligned} x - 2y &= 0 \\ x + y &= 3 \end{aligned} \iff A\vec{x} = \vec{b}$$

(*Hint. Clearly sketch and label the 3 known vectors. Now, how to combine 2 of them to get the third?*)



$x - 2y = 0$

$3y = 3$

$\therefore\ y = 1$

$\therefore\ x = 2$

so

$$2\vec{a}_1 + 1\cdot\vec{a}_2 = \vec{b}$$

8. Consider the linear system

$$ax - 2y = 1$$
$$x + 4y = 3$$

(a) (*4 pts*) For which number a does elimination break down permanently, so that there are no solutions?

if $a = -\frac{1}{2}$

then permanent breakdown

$$-\tfrac{1}{2}x - 2y = 1$$
$$x + 4y = 3$$

$$-\tfrac{1}{2}x - 2y = 1$$
$R_2 \leftarrow R_2 + 2R_1$ $\quad 0x + 0y = 5$ ← no soln

(b) (*4 pts*) For which number a does elimination break down temporarily, so that a row swap allows a solution?

if $a = 0$ then system

$$-2y = 1$$
$$x + 4y = 3$$

requires row swap for a solution (by back-subst.)

(c) (*3 pts*) For the value of a found in part **(b)**, solve the system.

after swap: $\quad x + 4y = 3$
$\quad -2y = 1$

back-subst. $\quad y = -\frac{1}{2}$

$$x = \frac{3 - 4(-\frac{1}{2})}{1} = 5$$

Extra Credit. (*3 pts*) Find the quadratic polynomial $p(x) = a + bx + cx^2$ which passes through the points $(-1, 1)$, $(1, 5)$, $(3, 17)$.

set-up linear system for coefficients and solve:

$$\begin{aligned} a - b + c &= 1 \\ a + b + c &= 5 \\ a + 3b + 9c &= 17 \end{aligned} \quad \longrightarrow \quad \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 4 \\ 0 & 4 & 8 & 16 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & 8 & 8 \end{bmatrix}$$

$a = 2$
$b = 2$
$c = 1$

9. (*6 pts*) Suppose $L = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$. What is L^{-1}?

do Gauss-Jordan, but only the elimination part:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ a & 1 & 0 & 0 & 1 & 0 \\ b & c & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -a & 1 & 0 \\ 0 & c & 1 & -b & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -a & 1 & 0 \\ 0 & 0 & 1 & -b+ac & -c & 1 \end{array}\right]$$

so

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{bmatrix}$$

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