Name: ____________________________________

Math 314 Linear Algebra (Bueler) Friday, 29 April 2022

Final Exam

No book, electronics, calculator, or internet access. Allowed notes: one sheet of letter paper (= $8.5'' \times 11''$ paper), with anything written on both sides. 150 points possible. 125 minutes maximum.

1. Consider this 3×5 matrix and its row-reduced echelon form ($R = \texttt{rref(A)}$):

$$A = \begin{bmatrix} 0 & 1 & 2 & 0 & -1 \\ 0 & 2 & 4 & 0 & -2 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \qquad \to \qquad R = \begin{bmatrix} 0 & 1 & 0 & -2 & -5 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

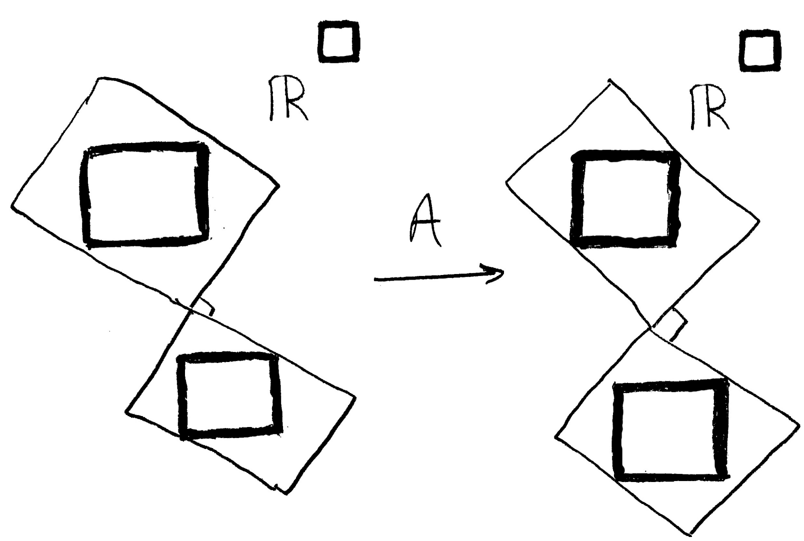
(a) (*3 pts*) What is the rank of A?

(b) (*3 pts*) What is the dimension of the null space of A?

(c) (*7 pts*) Find a basis for each of these 3 subspaces associated to A:

row space $C(A^\top)$, column space $C(A)$, null space $N(A)$

(d) (*5 pts*)
Fill-in the **6 bold boxes** in the "big picture" at right. Give either the name of the subspace or the dimension which applies to the specific matrix A:



2. Suppose A is any m by n matrix.

(a) (*7 pts*) Show that the null space $N(A)$ is a subspace.

(b) (*7 pts*) Show that the null space $N(A)$ and the row space $C(A^\top)$ are orthogonal.

3. (*7 pts*) Find the general solution of this linear system of two equations, and show your work:

$$\begin{aligned} 3x_1 - x_2 &= 4 \\ -6x_1 + 2x_2 &= -8 \end{aligned}$$

4. **(a)** (*10 pts*) Use the Gauss-Jordan elimination to compute the inverse of

$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -2 & -6 \\ 1 & 4 & 4 \end{bmatrix}$. Show your work.

(b) (*5 pts*) At the first stages of elimination in part **(a)** you applied row operations which can be written as elimination matrices. Specifically, give the matrices E_{21} and E_{31} which you used.

5. Is each statement about square matrices **True** or **False**? **If False, provide a counterexample.**

(a) (*3 pts*) If Q is an orthogonal matrix then Q is invertible.

(b) (*3 pts*) If P is a projection matrix then P is invertible.

(c) (*3 pts*) If P is a permutation matrix then P is invertible.

(d) (*3 pts*) If A is a diagonalizable matrix then A is invertible.

(e) (*3 pts*) If all eigenvalues of a matrix A are zero then $A = 0$.

(f) (*3 pts*) If any eigenvalue of A is zero then A is not invertible.

(g) (*3 pts*) If $A^2 = 0$ then $A = 0$.

6. (a) (*10 pts*) Form and solve the normal equations:. Show your work.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 2 & 0 \\ 2 & -1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

(b) (*4 pts*) In part **(a)**, is the vector $\mathbf{b}$ in the subspace $C(A)$? Explain your reasoning.

7. (*7 pts*) Recall that the projection matrix P which projects onto the column space $C(A)$ of a matrix A with linearly-independent columns (full column rank) is $P = A(A^\top A)^{-1}A^\top$. Show that P is a projection matrix.

8. **(a)** (*7 pts*) Suppose that a square matrix B is diagonalizable, that is, suppose $B = X\Lambda X^{-1}$ where X is invertible and Λ is diagonal. Give a formula which shows why it is easy to compute B^{100}.

(b) (*7 pts*) Describe how to compute $\det(B)$ if $B = X\Lambda X^{-1}$ is diagonalized.

9. (*7 pts*) Let $Q = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$. Show that Q is an orthogonal matrix.

10. **(a)** (*5 pts*) Recall that 2 by 2 rotation matrices have the form

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Using this form, find a non-identity matrix A with the property that $A^4 = I$, and verify this property for your matrix.

(b) (*5 pts*) Compute the eigenvalues of the matrix A which you found in part **(a)**.

11. **(a)** (*10 pts*) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

(b) (*3 pts*) Is the matrix in part **(a)** diagonalizable? Give a brief justification. (*Hint.* No further calculations are necessary.)

12. Consider the transformations from $\mathbf{V} = \mathbb{R}^2$ to $\mathbf{W} = \mathbb{R}^2$. For each one, is it linear? (*Show it is, or give a counterexample.*) Then give a simplified formula for $T(T(\mathbf{v}))$.

(a) (*5 pts*) $T(\mathbf{v}) = -\mathbf{v} + (1, 1)$

(b) (*5 pts*) $T(\mathbf{v}) = \frac{1}{2}(v_1 + v_2, v_1 + v_2)$

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Extra Credit. (*3 pts*) The matrix $Q = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$ in problem **8** is neither a rotation matrix nor a reflection matrix. But it can be factored into the product of a rotation matrix and a reflection matrix. Do so.

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