Name: SOLUTIONS

Math 314 Linear Algebra (Bueler) CORRECTED Friday, 29 April 2022

Final Exam

No book, electronics, calculator, or internet access. Allowed notes: one sheet of letter paper (= 8.5″ × 11″ paper), with anything written on both sides. 150 points possible. 125 minutes maximum.

1. Consider this 3×5 matrix and its row-reduced echelon form ($R = \texttt{rref(A)}$):

$$A = \begin{bmatrix} 0 & 1 & 2 & 0 & -1 \\ 0 & 2 & 4 & 0 & -2 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \qquad \to \qquad R = \begin{bmatrix} 0 & 1 & 0 & -2 & -5 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (*3 pts*) What is the rank of A? 2

(b) (*3 pts*) What is the dimension of the null space of A? 3

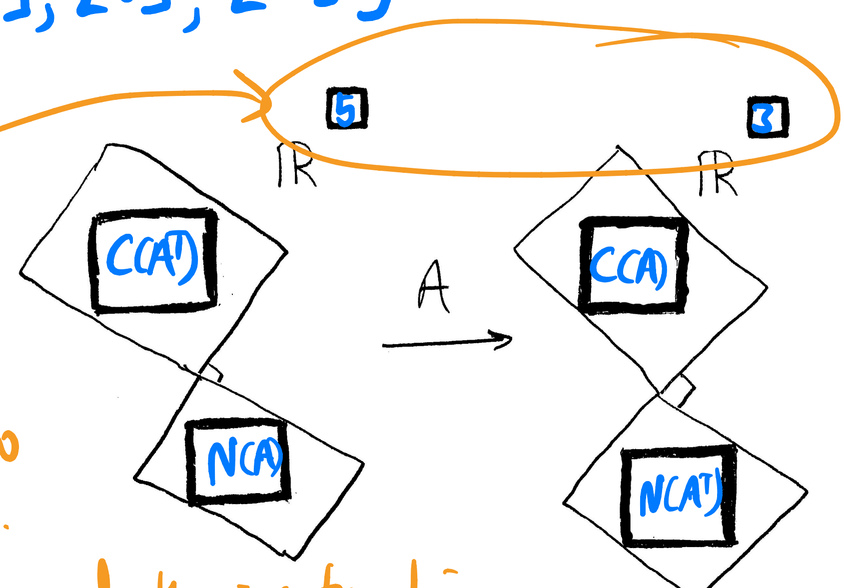
(c) (*7 pts*) Find a basis for each of these 3 subspaces associated to A:

row space $C(A^\top)$, column space $C(A)$, null space $N(A)$

$$C(A^T) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}, \qquad C(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \right\},$$

$$N(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

(d) (*5 pts*) Fill-in the **6 bold boxes** in the "big picture" at right. Give either the name of the subspace or the dimension which applies to the specific matrix A:



I decided to ignor these.

No one read the instructions.

2. Suppose A is any m by n matrix.

(a) (*7 pts*) Show that the null space $N(A)$ is a subspace.

If $\vec{x}$ and $\vec{y}$ are in $N(A)$, and c, d are real numbers, then $A(c\vec{x}+d\vec{y}) = cA\vec{x} + dA\vec{y} = c\cdot\vec{0}+d\cdot\vec{0}=\vec{0}$.

Thus $c\vec{x}+d\vec{y}$ is in $N(A)$, so $N(A)$ is a subspace.

(b) (*7 pts*) Show that the null space $N(A)$ and the row space $C(A^\top)$ are orthogonal.

If $\vec{x}$ is in $N(A)$ and $\vec{y} = A^T\vec{w}$ is in $C(A^T)$ then $\vec{x}^T\vec{y} = \vec{x}^T A^T \vec{w} = (A\vec{x})^T\vec{w} = \vec{0}^T\vec{w} = 0$.

Thus $\vec{x}, \vec{y}$ are orthogonal, so $N(A), C(A^T)$ are orthogonal.

3. (*7 pts*) Find the general solution of this linear system of two equations, and show your work:

$$3x_1 - x_2 = 4$$
$$-6x_1 + 2x_2 = -8$$

$R_2 \leftarrow R_2 + 2R_1$ $\quad \left[\begin{array}{cc|c} 3 & -1 & 4 \\ 0 & 0 & 0 \end{array}\right]$

$$\left.\begin{array}{l} \vec{x}_p = \begin{bmatrix} 4/3 \\ 0 \end{bmatrix} \\ \vec{s}_1 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \end{array}\right\} \quad \vec{x} = \vec{x}_p + t\vec{s}_1 = \begin{bmatrix} 4/3 + 1/3\,t \\ t \end{bmatrix}$$

set free variable x_2 to 0 for $\vec{x}_p$, 1 for $\vec{s}_1$

4. **(a)** (*10 pts*) Use the Gauss-Jordan elimination to compute the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & -2 & -6 \\ 1 & 4 & 4 \end{bmatrix}.$$ Show your work.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ -2 & -2 & -6 & 0 & 1 & 0 \\ 1 & 4 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1/2 & 0 \\ 0 & 1 & 1/2 & -1/2 & 0 & 1/2 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1/2 & -3/2 & -1/2 & 1/2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & -3 & -1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 2 & -3 \\ 0 & 1 & 0 & 1 & 1/2 & 0 \\ 0 & 0 & 1 & -3 & -1 & 1 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{= I} \qquad \underbrace{\begin{matrix} 8 & 2 & -3 \\ 1 & 1/2 & 0 \\ -3 & -1 & 1 \end{matrix}}_{= A^{-1}}$$

$$A^{-1} = \begin{bmatrix} 8 & 2 & -3 \\ 1 & 1/2 & 0 \\ -3 & -1 & 1 \end{bmatrix}$$

(b) (*5 pts*) At the first stages of elimination in part **(a)** you applied row operations which can be written as elimination matrices. Specifically, give the matrices E_{21} and E_{31} which you used.

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow \text{used as: } [E_{21}A \mid E_{21}I]$$

$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \leftarrow \text{then: } [E_{31}E_{21}A \mid E_{31}E_{21}I]$$

etc.

5. Is each statement about square matrices **True** or **False**? **If False, provide a counterexample.**

(a) (*3 pts*) If Q is an orthogonal matrix then Q is invertible.

true

($Q^T Q = I$ so $Q^{-1} = Q^T$)

(b) (*3 pts*) If P is a projection matrix then P is invertible.

false

$P = 0$ satisfies $P^2 = P$

(c) (*3 pts*) If P is a permutation matrix then P is invertible.

true

(P has $\det(P) = \pm 1$, for example)

(d) (*3 pts*) If A is a diagonalizable matrix then A is invertible.

false

$A = 0$ is diagonalizable: $0 = I0I^{-1}$

(e) (*3 pts*) If all eigenvalues of a matrix A are zero then $A = 0$.

false

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

(f) (*3 pts*) If any eigenvalue of A is zero then A is not invertible.

true

(if $A\vec{v} = 0\vec{v} = \vec{0}$ for $\vec{v} \neq 0$ then $N(A) \neq Z$)

(g) (*3 pts*) If $A^2 = 0$ then $A = 0$.

false

$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

6. **(a)** (*10 pts*) Form and solve the normal equations:. Show your work.

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 2 & 0 \\ 2 & -1 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$A^T A \vec{x} = A^T \vec{b} \quad \Big\} \text{ normal eqns}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 2 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 10 & -3 \\ -3 & 2 \end{bmatrix}$$

$$A^T \vec{b} = \begin{bmatrix} 1 & 1 & 2 & 2 \\ -1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 10 & -3 & 3 \\ -3 & 2 & -1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 10 & -3 & 3 \\ 0 & 11/10 & -1/10 \end{array}\right] \qquad R_2 \Leftarrow R_2 + \tfrac{3}{10} R_1$$

$$x_2 = (-1/10)/(11/10) = -1/11$$

$$10 x_1 - 3(-1/11) = 3 \Leftrightarrow 10 x_1 = 3 - 3/11 \Leftrightarrow x_1 = \left(\tfrac{30}{11}\right)/10 = 3/11$$

$$\vec{x} = \begin{bmatrix} 3/11 \\ -1/11 \end{bmatrix}$$

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(b) (*4 pts*) In part **(a)**, is the vector **b** in the subspace $C(A)$? Explain your reasoning.

<u>no.</u> if $\vec{b}$ were in $C(A)$ then $A\vec{x} = \vec{b}$ exactly because normal equations are $A\vec{x} = P\vec{b}$ and $P\vec{b} = \vec{b}$ if $\vec{b}$ in $C(A)$. but $A\vec{x} = \begin{bmatrix} 4/11 \\ 3/11 \\ 6/11 \\ 7/11 \end{bmatrix} \neq \vec{b}$

7. (*7 pts*) Recall that the projection matrix P which projects onto the column space $C(A)$ of a matrix A with linearly-independent columns (full column rank) is $P = A(A^\top A)^{-1}A^\top$. Show that P is a projection matrix.

$$P^2 = A(A^TA)^{-1}\underbrace{A^TA(A^TA)^{-1}}_{=I}A^T = A(A^TA)^{-1}A^T = P$$

so P is a projection

8. **(a)** (*7 pts*) Suppose that a square matrix B is diagonalizable, that is, suppose $B = X\Lambda X^{-1}$ where X is invertible and Λ is diagonal. Give a formula which shows why it is easy to compute B^{100}.

$$B^{100} = X\Lambda \underbrace{X^{-1}X}_{I}\Lambda \underbrace{X^{-1}\cdots X}_{I}\Lambda X^{-1}$$

$$= X\Lambda^{100}X^{-1}$$

easy: $\Lambda^{100} = \begin{bmatrix} \lambda_1^{100} & & \\ & \ddots & \\ & & \lambda_n^{100} \end{bmatrix}$

(b) (*7 pts*) Describe how to compute $\det(B)$ if $B = X\Lambda X^{-1}$ is diagonalized.

$$\det(B) = \det(X)\det(\Lambda)\det(X^{-1})$$

$$= \det(X)(\lambda_1 \cdots \lambda_n)\frac{1}{\det(X)}$$

$$= \lambda_1 \cdots \lambda_n = \prod_{i=1}^{n} \lambda_i$$

9. (*7 pts*) Let $Q = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$. Show that Q is an orthogonal matrix.

$$Q^TQ = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1+3 & -\sqrt{3}+\sqrt{3} \\ -\sqrt{3}+\sqrt{3} & 1+3 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

10. **(a)** (*5 pts*) Recall that 2 by 2 rotation matrices have the form

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Using this form, find a non-identity matrix A with the property that $A^4 = I$, and verify this property for your matrix.

$\theta = \pi/2$: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\therefore A^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $A^4 = (A^2)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$ ✓

(b) (*5 pts*) Compute the eigenvalues of the matrix A which you found in part **(a)**.

$$p(\lambda) = \det\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

$$\lambda_1 = +i, \quad \lambda_2 = -i$$

11. **(a)** (*10 pts*) Find the eigenvalues and eigenvectors of the matrix
$$A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}.$$

$$p(\lambda) = \det \begin{bmatrix} 2-\lambda & 2 & 2 \\ 2 & -\lambda & 0 \\ 2 & 0 & -\lambda \end{bmatrix}$$

$$= (2-\lambda)(\lambda^2 - 0) - 2(-2\lambda - 0) + 2(0 + 2\lambda)$$

$$= (2-\lambda)\lambda^2 + 4\lambda + 4\lambda = \lambda(2\lambda - \lambda^2 + 8)$$

$$= -\lambda(\lambda^2 - 2\lambda - 8) = -\lambda(\lambda-4)(\lambda+2) \quad \therefore \lambda = -2, 0, 4$$

$\lambda_1 = -2$: $\begin{bmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \vec{x} = \vec{0} \rightarrow \begin{bmatrix} 4 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \vec{x} = \vec{0} \rightarrow \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$ $\quad \vec{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda_2 = 0$: $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \vec{x} = \vec{0} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$ $\quad \vec{x}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$\lambda_3 = 4$: $\begin{bmatrix} -2 & 2 & 2 \\ 2 & -4 & 0 \\ 2 & 0 & -4 \end{bmatrix} \vec{x} = \vec{0} \rightarrow \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$ $\quad \vec{x}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

(b) (*3 pts*) Is the matrix in part **(a)** diagonalizable? Give a brief justification. (*Hint.* No further calculations are necessary.)

yes. A is symmetric

12. Consider the transformations from $\mathbf{V} = \mathbb{R}^2$ to $\mathbf{W} = \mathbb{R}^2$. For each one, is it linear? (*Show it is, or give a counterexample.*) Then give a simplified formula for $T(T(\mathbf{v}))$.

(a) (*5 pts*) $T(\mathbf{v}) = -\mathbf{v} + (1,1)$

not linear: $T(\vec{0}) = (1,1) \neq \vec{0}$

$T(T(\vec{v})) = T(-\vec{v} + (1,1)) = -(-\vec{v}+(1,1)) + (1,1)$

$= \vec{v} - (1,1) + (1,1) = \vec{v}$

(b) (*5 pts*) $T(\mathbf{v}) = \frac{1}{2}(v_1 + v_2, v_1 + v_2)$

linear: $T(c\vec{v}+d\vec{w}) = T((av_1+dw_1, av_2+dw_2))$

$= \frac{1}{2}(av_1+dw_1+av_2+dw_2, av_1+dw_1+av_2+dw_2)$

$= \frac{1}{2}(av_1+av_2, av_1+av_2) + \frac{1}{2}(dw_1+dw_2, dw_1+dw_2)$

$= a\cdot\frac{1}{2}(v_1+v_2, v_1+v_2) + d\cdot\frac{1}{2}(w_1+w_2, w_1+w_2)$

$= aT(\vec{v}) + dT(\vec{w})$

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$T(T(\vec{v})) = T\left(\left(\frac{v_1+v_2}{2}, \frac{v_1+v_2}{2}\right)\right)$

T is a projection

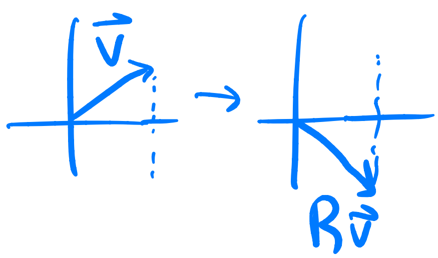
$= \frac{1}{2}\left(\frac{v_1+v_2}{2} + \frac{v_1+v_2}{2}, \frac{v_1+v_2}{2} + \frac{v_1+v_2}{2}\right)$

$= \frac{1}{2}(v_1+v_2, v_1+v_2) = T(\vec{v})$

Extra Credit. (*3 pts*) The matrix $Q = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix}$ in problem $\cancel{2}$ 9 (sorry) is neither a rotation matrix nor a reflection matrix. But it can be factored into the product of a rotation matrix and a reflection matrix. Do so.

$$Q = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}\underbrace{\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}}$$

R = reflection across x-axis:



S = rotation by $\pi/3$ radians

$$= \begin{bmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ \sin(\pi/3) & \cos(\pi/3) \end{bmatrix}$$

see problem 10

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