SOLUTIONS

Some High Points of Calculus

This is a review of some calculus you will need for differential equations.

recall: $(\tan x)' = \sec^2 x$

1. *Chain rule.* Recall the chain rule

$$[f(g(x))]' = f'(g(x))\, g'(x)$$

(a) Identify an outer function $f(x)$ and an inner function $g(x)$ if $f(g(x)) = \sqrt{\tan x + 3x}$. Then compute the following derivative.

$f'(x)$

$$\left[\sqrt{\tan x + 3x}\right]' = \frac{1}{2}\underbrace{(\tan x + 3x)}_{g(x)\text{ unchanged}}^{-1/2}\underbrace{(\sec^2 x + 3)}_{g'(x)}$$

$f(x) = \sqrt{x}$

$g(x) = \tan x + 3x$

essentially the only choice in this case

(b) Construct your own chain rule example:

$f(x) =$ e^x

$g(x) =$ $\sin x$

$[f(g(x))]' =$ $e^{\sin x} \cos x$

$f(g(x)) = e^{\sin x}$

in Matlab:

e^x is

`>> exp(x)`

$f(g(x))$ is

`>> exp(sin(x))`

(*Make it different from* **(a)** *and non-trivial but not too complicated. In particular, neither* $f(x)$ *nor* $g(x)$ *should be as simple as a linear function, i.e.* $ax + b$*. Remember that* $x^k, e^x, \ln x, b^x$, $\log_b x, \sin x, \cos x, \tan x, \sec x, \arcsin x$ *are common functions from calculus which you must be able to correctly differentiate! Use this problem to practice those that are least familiar?*)

read as "antiderivative of $\frac{1}{x}$"

2. *Indefinite integration.* Remember that the indefinite integral just means "anti-derivative," so $\int f(x)\,dx = F(x) + C$ means exactly the same thing as $(F(x))' = f(x)$. Recall you can do some integrals just by recognizing a derivative, perhaps with some fiddling with constants.

(a) Compute the indefinite integral:

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

note: domain of $\ln(x)$ is $(0,+\infty)$

domain: $(-\infty,0)\cup(0,\infty)$ ← same

domain: $(-\infty,0)\cup(0,\infty) = \{x \neq 0\}$

(b) Compute the indefinite integral:

$$\int 3^{2x}\,dx = \frac{3^{2x}}{2\ln 3} + C$$

Recall: $(a^x)' = (\ln a)\,a^x$

3. *Integration by substitution.* Substitution is **the chain rule in reverse**. For example, from **1 (a)** we have

$$\int \frac{\sec^2 x + 3}{\sqrt{\tan x + 3x}}\,dx = \int \frac{du}{\sqrt{u}} \qquad [\text{with } u = \tan x + 3x]$$
$$= 2u^{1/2} + C = 2\sqrt{\tan x + 3x} + C$$

(It is common to need to fiddle with constant factors like the "2" here.) In general:

$$\int f'(g(x))g'(x)\,dx = \int f'(u)\,du = f(u) + C = f(g(x)) + C$$

(a) Turn your chain rule example in **1 (b)** into an integration by substitution.

$f'(x) = e^x$ $\quad g(x) = \sin x$ $\quad g'(x) = \cos x$

$u = \sin x$

$du = \cos x\,dx$

$$\int e^{\sin x}\cos x\,dx = \int e^u\,du$$
$$= e^u + C$$
$$= e^{\sin x} + C$$

4. *Product rule and integration-by-parts.* The product rule

$$[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x)$$

can be used in reverse too. The indefinite integral of both sides of the above gives

$$u(x)v(x) = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx.$$

The main use of this is to exchange one of the last two integrals for the other, which is integration-by-parts:

$$\int u(x)v'(x)\,dx = u(x)v(x) - \int u'(x)v(x)\,dx.$$

(You probably have this memorized as $\int u\,dv = uv - \int v\,du$.)

(a) Construct your own product rule example:

$u = x^2$, $v = e^x$ | $u = x$, $v = \ln x$ | $u = e^x$, $v = \sin x$

$u(x) =$ $\sin(x^2)$

$v(x) =$ $\ln x$

$[u(x)v(x)]' =$ $(\sin(x^2)\cdot\ln x)' = \cos(x^2)(2x)\cdot\ln x + \sin(x^2)\cdot\frac{1}{x}$

(*Again, make it non-trivial but not too complicated.*)

(b) Turn the above example into an integration-by-parts example.

$$\int \sin(x^2)\,\frac{1}{x}\,dx = \sin(x^2)\ln x - \int \ln x \cos(x^2)(2x)\,dx$$

5. *Fundamental Theorem of Calculus (FTC).* When you compute a definite integral by hand you usually use a form of the FTC:

FTC II

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad \text{where } F'(x) = f(x)$$

This says that doing an integral is the same as un-doing a derivative.

Recall that if you do a definite integral by substitution then you can change the limits:

$$\int_{x=a}^{x=b} f(g(x))\,g'(x)\,dx = \int_{u=g(a)}^{u=g(b)} f(u)\,du = F(g(b)) - F(g(a))$$

(a) Compute

$u = \sin x$, $du = \cos x$

$w = 3u$, $dw = 3du$, $du = \frac{dw}{3}$

$$\int_{\pi/6}^{\pi/2} \frac{\cos x}{1 + 9\sin^2 x}\,dx = \int_{1/2}^{1} \frac{du}{1+9u^2} = \int_{3/2}^{3} \frac{dw/3}{1+w^2} = \frac{1}{3}\int_{3/2}^{3} \frac{dw}{1+w^2}$$

FTC II

$$= \frac{1}{3} \arctan w \Big]_{3/2}^{3} = \boxed{\frac{1}{3}\left(\arctan(3) - \arctan(3/2)\right)}$$

There is another form of the FTC, often called "FTC I." It says that a derivative un-does an integral:

$$\frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$$

The integral inside the parentheses computes the area under the curve $y = f(t)$ from $t = a$ to $t = x$. One should think of this area varying as x changes, thus defining a function $g(x) = \int_a^x f(t)\,dt$. One may answer some questions about $g(x)$ even when an antiderivative of the integrand is not known.

(b) Suppose we define

$$g(x) = \int_2^x \sin(e^t)\,dt$$

Compute the exact value of $g(2)$.

$g(2) = 0$

(c) For the same function $g(x)$, find $g'(x)$.

$g'(x) = \sin(e^x)$ } FTC I is easy!