8.4 The matrix exponential solves systems

a lecture for MATH F302 Differential Equations

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for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

solving the simplest ODEs

- *simplest scalar ODE:*

$$y' = ay \qquad \text{has solution} \qquad y(t) = ce^{at}$$

- *simplest system of ODEs:*

$$\mathbf{X}' = \mathbf{A}\mathbf{X} \qquad \text{has solution} \qquad \mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C}$$

  - the last formula is new in §8.4

what does $e^{\mathbf{A}t}$ mean?

- what does $e^{\mathbf{A}t}$ mean?
  - what does e^{at} mean?
    - what does e^x mean? $\longleftarrow$ we know this one!

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

- *definition.* if $\mathbf{A}$ is a square matrix and t is any number then

$$\begin{aligned} e^{\mathbf{A}t} &= \mathbf{I} + \mathbf{A}t + \mathbf{A}^2 \frac{t^2}{2} + \mathbf{A}^3 \frac{t^3}{3!} + \mathbf{A}^4 \frac{t^4}{4!} + \dots \\ &= \sum_{k=0}^{\infty} \mathbf{A}^k \frac{t^k}{k!} \end{aligned}$$

  - note $\mathbf{A}^0 = \mathbf{I}$ makes sense if we believe $x^0 = 1$
  - also recall $0! = 1$

like an exercise on Homework §8.4

- $e^{\mathbf{A}t} = \mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \mathbf{A}^3\frac{t^3}{3!} + \mathbf{A}^4\frac{t^4}{4!} + \dots$
- *example 1.* use the above series to compute $e^{\mathbf{A}t}$ and $e^{-\mathbf{A}t}$, in simplified form, if

$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$$

like an exercise on Homework §8.4

- *example 2.* use the series definition to compute $e^{\mathbf{A}t}$, in simplified form, if

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & 3 & 0 \end{pmatrix}$$

like an exercise on Homework §8.4

- *example 3*. use the series definition to compute $e^{\mathbf{A}t}$, in simplified form, if

$$\mathbf{A} = \begin{pmatrix} -3 & -3 & -3 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{pmatrix}$$

last two were special

- *example 1* illustrates that *diagonal* matrices are easy to exponentiate
- *example 2* and *example 3* had unusual matrices with the property that some power was the zero matrix:

$$\mathbf{A}^k = \mathbf{0}$$

- but the above are not typical cases
- for most $\mathbf{A}$:
  - $e^{\mathbf{A}t}$ has infinitely-many nonzero terms
  - the pattern is harder to see

like an exercise on Homework §8.4

- *example 4*. use the series definition to compute $e^{\mathbf{A}t}$, in simplified form, if

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

derivative of $e^{\mathbf{A}t}$

- *fact 1.*

$$\frac{d}{dt}\left(e^{\mathbf{A}t}\right) = \mathbf{A}e^{\mathbf{A}t}$$

Proof.

$$\begin{aligned}\frac{d}{dt}\left(e^{\mathbf{A}t}\right) &= \frac{d}{dt}\left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \mathbf{A}^3\frac{t^3}{3!} + \mathbf{A}^4\frac{t^4}{4!} + \dots\right) \\ &= \mathbf{0} + \mathbf{A} + \mathbf{A}^2\frac{2t}{2} + \mathbf{A}^3\frac{3t^2}{3!} + \mathbf{A}^4\frac{4t^3}{4!} + \dots \\ &= \mathbf{A}\left(\mathbf{I} + \mathbf{A}t + \mathbf{A}^2\frac{t^2}{2} + \mathbf{A}^3\frac{t^3}{3!} + \dots\right) = \mathbf{A}e^{\mathbf{A}t}\end{aligned}$$

□

- *fact 2.* if $\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C}$ then $\mathbf{X}' = \mathbf{A}\mathbf{X}$
- *fact 3.* if $\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C}$ then $\mathbf{X}(0) = \mathbf{C}$

the matrix exponential solves systems

in summary:

1. for the ODE

$$\mathbf{X}' = \mathbf{A}\mathbf{X}$$

   the general solution is

$$\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C}$$

   where $\mathbf{C} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$ is a vector of unknown constants

2. for the ODE IVP

$$\mathbf{X}' = \mathbf{A}\mathbf{X}, \qquad \mathbf{X}(0) = \mathbf{X}_0$$

   the solution is

$$\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{X}_0$$

use a computer ...

- for the ODE IVP

$$\mathbf{X}' = \mathbf{A}\mathbf{X}, \qquad \mathbf{X}(0) = \mathbf{C}$$

suppose you want the solution at time t:

$$\mathbf{X}(T) = e^{\mathbf{A}t}\mathbf{C}$$

- with Matlab or Octave:

```
>> A = [...];               % enter square matrix A
>> C = [...];               % enter column vector C
>> t = ...                  % set the time
>> expm(A*t) * C            % bam! done!
```

- `expm()` computes the matrix exponential
  - be careful ... `exp()` is *not* what you want

... for fast numbers

- *example 5*. solve the initial value problem for $x(2), y(2), z(2)$:

$$\begin{aligned} x' &= 2x - 5y + z & x(0) &= -2 \\ y' &= -x + y + 3z & y(0) &= 0 \\ z' &= x - 2y - z & z(0) &= 3 \end{aligned}$$

solution.

```
>> A = [2 -5 1; -1 1 3; 1 -2 -1];
>> C = [-2; 0; 3];
>> expm(A*2) * C
ans =
     -1227.9
      68.564
     -381.69
```

so: $x(2) = -1227.9$, $y(2) = 68.564$, $z(2) = -381.69$

can you check it?

- what tool would help us quickly check previous slide result?
  - consider all the tools in the whole course ...
- *answer.* the most general tool is numerical approximation

- *example 5, cont.* check the solution on the previous slide

solution.

```
>> f = @(t,U) [2*U(1)-5*U(2)+U(3); ...
               -U(1)+U(2)+3*U(3); ...
               U(1)-2*U(2)-U(3)];
>> [tt,UU] = ode45(f,[0,2],[-2;0;3]);
>> UU(end,:)
ans =
     -1227.9       68.565       -381.7
```

by-hand usage?

- as long as we can compute $e^{\mathbf{A}t}$ by hand, we can solve an ODE by hand using the matrix exponential
- *example 6*. use $\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C}$ to find the general solution of the given system

$$\mathbf{X}' = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ -1 & 3 & 0 \end{pmatrix} \mathbf{X}$$

expectations

- to learn this material, just listening to a lecture is *not* enough
  - *read* section 8.4
  - *read* and do Homework 8.4
- I hope you have learned something from this course!
  - perhaps even found it interesting?
  - note that
    - MATH 302 discrete mathematics
    - MATH 314 linear algebra
    - STAT 300 probability and statistics
    - MATH 265 introduction to mathematical proof

    are courses of more-or-less comparable level to MATH 302
  - a mathematics minor requires calculus I,II,III ✓, and three MATH courses at the 300/400 level (including MATH 265 and STAT 300 as special cases) . . . ✓ on one of those!