7.3 Laplace Transforms: translations & unit step functions

a lecture for MATH F302 Differential Equations

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for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

the Laplace transform strategy

$$
\begin{tikzcd}
\boxed{\text{ODE IVP for } y(t)} \arrow[r, dashed, "\textit{direct}"] \arrow[d, "\mathcal{L}"'] & \boxed{y(t) = \dots} \\
\boxed{\begin{array}{c}\text{algebraic equation} \\ \text{for } Y(s)\end{array}} \arrow[r, "\text{solve for } Y"] & \boxed{Y(s) = \dots} \arrow[u, "\mathcal{L}^{-1}"']
\end{tikzcd}
$$

- need §7.3: "operational properties" esp. translations (shifts)
  - including the *unit step function* $\mathcal{U}(t)$

recall Laplace's Transform

- the definition:

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

- when applying $\mathcal{L}$ to an ODE use:

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$$
$$\mathcal{L}\{y''(t)\} = s^2 Y(s) - sy(0) - y'(0)$$

- doing $\mathcal{L}^{-1}$ is mostly use of a table, e.g.:
  - $\mathcal{L}^{-1}\left\{\dfrac{1}{s-4}\right\} = e^{4t}$
  - $\mathcal{L}^{-1}\left\{\dfrac{k}{(s-a)^2+k^2}\right\} = e^{at}\sin kt$

Pierre-Simon Laplace (1749–1827)

we have a decent table

TABLE OF LAPLACE TRANSFORMS:

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\left\{t^{-1/2}\right\} = \frac{\sqrt{\pi}}{s^{1/2}}$$

$$\mathcal{L}\left\{t^{1/2}\right\} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

$$\mathcal{L}\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$$

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$$

$$\mathcal{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2}$$

$$\mathcal{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2}$$

$$\mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}$$

$$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}\{e^{at}\sin(kt)\} = \frac{k}{(s-a)^2+k^2}$$

$$\mathcal{L}\{e^{at}\cos(kt)\} = \frac{s-a}{(s-a)^2+k^2}$$

$$\mathcal{L}\{t\sin(kt)\} = \frac{2ks}{(s^2+k^2)^2}$$

$$\mathcal{L}\{t\cos(kt)\} = \frac{s^2-k^2}{(s^2+k^2)^2}$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}$$

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$$

$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$$

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$$

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

$$\mathcal{L}\{f*g\} = F(s)G(s)$$

noticable in the table

- compare the left and right columns in this part of the table:

$$\mathcal{L}\{t\} = \frac{1}{s^2} \qquad \mathcal{L}\left\{te^{at}\right\} = \frac{1}{(s-a)^2}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \qquad \mathcal{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2} \qquad \mathcal{L}\left\{e^{at}\sin(kt)\right\} = \frac{k}{(s-a)^2+k^2}$$

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2} \qquad \mathcal{L}\left\{e^{at}\cos(kt)\right\} = \frac{s-a}{(s-a)^2+k^2}$$

- multiplying by e^{at} causes: $\quad s \to s-a$
- this is a rule!: multiplying by an exponential in t is *translation* in s:

$$\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)$$

why?

- *why* does multiplying by e^{at} cause $\;s \to s - a\;$?
- recall definition:

$$\mathcal{L}\left\{f(t)\right\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

- so:

$$\begin{aligned}\mathcal{L}\left\{e^{at} f(t)\right\} &= \int_0^\infty e^{-st} e^{at} f(t)\, dt = \int_0^\infty e^{-(s-a)t} f(t)\, dt \\ &= F(s-a)\end{aligned}$$

examples from §7.3

- start by just going back and forth using the new rule
- *exercise 1.*

  $$\mathcal{L}\left\{e^{2t}\sin(3t)\right\} =$$

- *exercise 2.*

  $$\mathcal{L}^{-1}\left\{\frac{1}{s^2-6s+10}\right\} =$$

example like §7.3 #23

- *exercise 3.* use $\mathcal{L}$ to solve the ODE IVP:

  $$y''+4y'+4y = 0, \quad y(0) = 1,\ y'(0) = 1$$

example like §7.3 #30

- *exercise 4.* use $\mathcal{L}$ to solve the ODE IVP:

  $$y'' - 2y' + 5y = t, \quad y(0) = 0,\ y'(0) = 7$$

unit step function

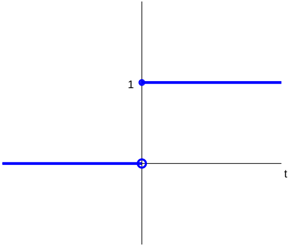
- *definition.* the *unit step function* is

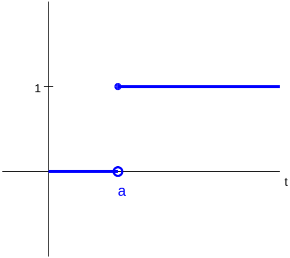
$$\mathcal{U}(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

- the book defines it with a translation, and only on $[0, \infty)$

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

- why? because we want to model "switching on" at time $t = a$





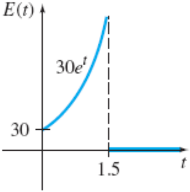
$\mathcal{U}(t-a)$ helps with switching on/off

write each function in terms of unit step function(s):

- *example A.*

$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t^2, & t \ge 1 \end{cases}$$

- *example B.*

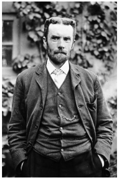


Laplace transform with $\mathcal{U}(t-a)$

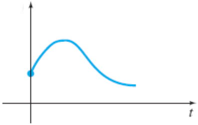
- $\mathcal{U}(t)$ is also called the *Heaviside* function
- easy-to-show: if $F(s) = \mathcal{L}\{f(t)\}$ then

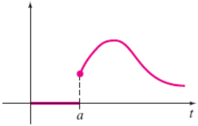
$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-at}F(s)$$

- *show it:*



Oliver Heaviside
(1850–1925)





#57 in §7.3

- *exercise 5.* write the function in terms of $\mathcal{U}$ and then find the Laplace transform:

$$f(t) = \begin{cases} 0, & 0 \leq t < 1 \\ t^2, & t \geq 1 \end{cases}$$

second version

- the book then says:
  We are frequently confronted with the problem of finding the Laplace transform of a product of a function g and a unit step function $\mathcal{U}(t-a)$ where the function g lacks the precise shifted form $f(t-a)$ in Theorem 7.3.2.

- yup, that's our problem
- 2nd form of the same rule:

$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-at}\mathcal{L}\{g(t+a)\}$$

- it will be in the table also, when it is printed on quizzes/exams

once again

- *exercise 5.* write the function in terms of $\mathcal{U}$ and then find the Laplace transform:

$$f(t) = \begin{cases} 0, & 0 \le t < 1 \\ t^2, & t \ge 1 \end{cases}$$

like #66 in §7.3

- *exercise 6.* use Laplace transforms to solve the ODE IVP:

$$y'' + 9y = f(t), \quad y(0) = 0,\ y'(0) = 0$$

where $\quad f(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & t \ge 1 \end{cases}$

summary

- assume $\mathcal{L}\{f(t)\} = F(s)$
- *1st translation theorem.*

$$\boxed{\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)}$$

- *2nd translation theorem.* if $a > 0$ then

$$\boxed{\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)}$$

  - includes easy case: $\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}$
  - second form

$$\boxed{\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}}$$

- these are all in the table you will get on quizzes and exams, so:
  goal is understanding not memorizing

expectations

to learn this material, just listening to a lecture is *not* enough

- *read* section 7.3
  - you can ignore "beams" and example 10 in §7.3
- find good youtube videos on Laplace transforms and inverse Laplace transforms?
- do Homework 7.3