7.2 *inverse* Laplace Transforms, and application to DEs

a lecture for MATH F302 Differential Equations

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for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

recall the definition

- the *Laplace transform* of a function $f(t)$ defined on $(0, \infty)$ is

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

  - this is well defined for $s > c$ if $f(t)$ has exponential order c: $|f(t)| \leq Me^{ct}$

- the result of applying the Laplace transform is a function of s:

$\mathcal{L}\{f(t)\} = \mathcal{L}\{f\}(s) = F(s)$ $\longleftarrow$ all mean the same

the Laplace transform strategy

$$\begin{tikzcd} \boxed{\text{ODE IVP for } y(t)} \arrow[r, dashed, "\textit{old method}"] \arrow[d, "\mathcal{L}"'] & \boxed{y(t) = \dots} \\ \boxed{\begin{array}{c}\text{algebraic equation}\\ \text{for } Y(s)\end{array}} \arrow[r, "\text{solve for } Y"] & \boxed{Y(s) = \dots} \arrow[u, "\mathcal{L}^{-1}"'] \end{tikzcd}$$

- §7.2: practice with $\mathcal{L}^{-1}$ then practice the whole strategy

bring a table to the party

Theorem 7.1.1 **Transforms of Some Basic Functions**

(a) $\mathscr{L}\{1\} = \dfrac{1}{s}$

(b) $\mathscr{L}\{t^n\} = \dfrac{n!}{s^{n+1}},\ n = 1, 2, 3, \ldots$

(c) $\mathscr{L}\{e^{at}\} = \dfrac{1}{s-a}$

(d) $\mathscr{L}\{\sin kt\} = \dfrac{k}{s^2 + k^2}$

(e) $\mathscr{L}\{\cos kt\} = \dfrac{s}{s^2 + k^2}$

(f) $\mathscr{L}\{\sinh kt\} = \dfrac{k}{s^2 - k^2}$

(g) $\mathscr{L}\{\cosh kt\} = \dfrac{s}{s^2 - k^2}$

- on page 282 of book
- this table is pathetic! better one soon ...

first $\mathcal{L}^{-1}$ example (like §7.2 #5)

- *exercise 1.* use algebra and a table of Laplace transforms:

$$\mathcal{L}^{-1}\left\{\frac{(s-1)^3}{s^4}\right\} =$$

$\mathcal{L}^{-1}$ example like §7.2 #11

- *exercise 2.* use algebra and a table of Laplace transforms:

$$\mathcal{L}^{-1}\left\{\frac{5}{s^2+36}\right\} =$$

$\mathcal{L}^{-1}$ example like §7.2 #18

- *exercise 3.* use algebra and a table of Laplace transforms:

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s^2-7s}\right\} =$$

not actually a better table

- compare Theorems 7.1.1 and 7.2.1
- they say the same thing!

Theorem 7.1.1 Transforms of Some Basic Functions

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(c) $\mathscr{L}\{e^{at}\} = \dfrac{1}{s-a}$

(d) $\mathscr{L}\{\sin kt\} = \dfrac{k}{s^2+k^2}$

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(g) $\mathscr{L}\{\cosh kt\} = \dfrac{s}{s^2-k^2}$

Theorem 7.2.1 Some Inverse Transforms

(a) $1 = \mathscr{L}^{-1}\left\{\dfrac{1}{s}\right\}$

(b) $t^n = \mathscr{L}^{-1}\left\{\dfrac{n!}{s^{n+1}}\right\}, n = 1, 2, 3, \ldots$

(c) $e^{at} = \mathscr{L}^{-1}\left\{\dfrac{1}{s-a}\right\}$

(d) $\sin kt = \mathscr{L}^{-1}\left\{\dfrac{k}{s^2+k^2}\right\}$

(e) $\cos kt = \mathscr{L}^{-1}\left\{\dfrac{s}{s^2+k^2}\right\}$

(f) $\sinh kt = \mathscr{L}^{-1}\left\{\dfrac{k}{s^2-k^2}\right\}$

(g) $\cosh kt = \mathscr{L}^{-1}\left\{\dfrac{s}{s^2-k^2}\right\}$

actually a decent table

Table of Laplace Transforms:

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\left\{t^{-1/2}\right\} = \frac{\sqrt{\pi}}{s^{1/2}}$$

$$\mathcal{L}\left\{t^{1/2}\right\} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

$$\mathcal{L}\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$$

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$$

$$\mathcal{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2}$$

$$\mathcal{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2}$$

$$\mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}$$

$$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}\{e^{at}\sin(kt)\} = \frac{k}{(s-a)^2+k^2}$$

$$\mathcal{L}\{e^{at}\cos(kt)\} = \frac{s-a}{(s-a)^2+k^2}$$

$$\mathcal{L}\{t\sin(kt)\} = \frac{2ks}{(s^2+k^2)^2}$$

$$\mathcal{L}\{t\cos(kt)\} = \frac{s^2-k^2}{(s^2+k^2)^2}$$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$$

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}$$

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$$

$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$$

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$$

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

$$\mathcal{L}\{f*g\} = F(s)G(s)$$

$\mathcal{L}^{-1}$ example like §7.2 #23

- *exercise 4.* use algebra and a table of Laplace transforms:

$$\mathcal{L}^{-1}\left\{\frac{s}{(s-3)(s-4)(s-6)}\right\} =$$

$$\frac{s}{(s-3)(s-4)(s-6)} = \frac{1}{s-3} - \frac{2}{s-4} + \frac{1}{s-6}$$

$\mathcal{L}^{-1}$ example like §7.2 #25

- *exercise 5.* use algebra and a table of Laplace transforms:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3+7s}\right\} =$$

transform of first derivatives

- *exercise 6.* suppose $F(s) = \mathcal{L}\{f(t)\}$. use the definition of the Laplace transform to show: $\mathcal{L}\{f'(t)\} = s\,F(s) - f(0)$

- actually we showed this on §7.1 slides
- what assumptions did we make about $f(t)$?

transform of second derivatives

- *exercise 7.* suppose $F(s) = \mathcal{L}\{f(t)\}$. show:

$$\mathcal{L}\left\{f''(t)\right\} = s^2 F(s) - s\,f(0) - f'(0)$$

- in the table you'll have in hand during quizzes/exams:

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$$

like §7.2 #39

- *exercise 8.* use Laplace transform to solve the ODE IVP:

$$y'' - 5y' + 4y = 0, \qquad y(0) = 1,\ y'(0) = 0$$

the old way

- *exercise 9.* solve without Laplace transform:

$$y'' - 5y' + 4y = 0, \qquad y(0) = 1,\ y'(0) = 0$$

like §7.2 #41

- *exercise 10.* use Laplace transform to solve the ODE IVP:

$$y'' + y = \sqrt{2}\cos(\sqrt{2}t), \qquad y(0) = 0,\ y'(0) = 3$$

like §7.2 #41, cont.

$$y(t) = 3\sin(t) + \sqrt{2}\cos(t) - \sqrt{2}\cos(\sqrt{2}t)$$

expectations

to learn this material, just listening to a lecture is *not* enough

- *read* section 7.2 (and 7.1 and 7.3)
- find good youtube videos on Laplace transforms and inverse Laplace transforms?
- do Homework 7.2