7.1 Laplace Transforms (starting from the definition)

a lecture for MATH F302 Differential Equations

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for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

the definition

- the Laplace transform of a function $f(t)$ defined on $(0, \infty)$ is

$$\boxed{\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt}$$

  - this is the book's notation

- the result of applying the Laplace transform is a function of s
  - so slightly better notation would be

$$\mathcal{L}\{f\}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

  - a common (and good) way to write it is

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

why do we use $\mathcal{L}$ in differential equations?

- why do we use

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt \quad ?$$

- because the Laplace transform converts a linear DE (in t) into an algebra problem (in s)
  - this is especially useful for solving *nonhomogeneous* DEs
  - . . . it's how many engineers think about nonhomogeneous DEs
  - the Laplace transform *linear*, that is,

$$\mathcal{L}\{f(t)+g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}, \text{ and}$$
$$\mathcal{L}\{\alpha f(t)\} = \alpha\mathcal{L}\{f(t)\}$$

  - the Laplace transform is basically limited to linear DEs

practice with integrals on $[a, \infty)$

- a Laplace transform is an integral $\int_0^\infty \dots$
- we need practice!
- *practice 1.* compute

$$\int_2^\infty e^{-3t}\,dt =$$

$$= \frac{1}{3}e^{-6}$$

- *practice 2.* compute and sketch

$$\int_1^\infty \frac{1}{t}\,dt =$$

$$= +\infty$$

practice integrals, cont.

- *practice 3*. compute and sketch

$$\int_0^{\infty} te^{-t}\,dt =$$

$$= +1$$

Laplace transforms, from the definition

- the technique in Chapter 7 requires pre-computing the Laplace transforms of some familiar functions, and then using these to solve DEs
- *example 1*. compute $\mathcal{L}\left\{e^{kt}\right\}$
- *example 2*. compute $\mathcal{L}\{1\}$

from the definition, cont.

- *example 3*. compute $\mathcal{L}\{t\}$

- *example 4*. compute $\mathcal{L}\{\cos(kt)\}$

from the definition, cont.[2]

- *example 5*. compute $\mathcal{L}\{t^n\}$

$$\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\left\{t^{n-1}\right\}$$

$$\implies \mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

first table

Theorem 7.1.1 **Transforms of Some Basic Functions**

(a) $\mathscr{L}\{1\} = \frac{1}{s}$

(b) $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, n = 1, 2, 3, \ldots$

(c) $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}$

(d) $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}$

(e) $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}$

(f) $\mathscr{L}\{\sinh kt\} = \frac{k}{s^2-k^2}$

(g) $\mathscr{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$

- you will have a table like this on quizzes and exams
- *and* it is a fair question to ask you to show any one from the definition

piecewise functions

- *example 6*. compute $\mathcal{L}\{f\}$ if

$$f(t) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t > a \end{cases}$$

$$\mathcal{L}\{f\} = \frac{e^{-as}}{s}$$

key fact from §7.2

- *example 7*. let $Y(s) = \mathcal{L}\{y(t)\}$. use the definition to show

$$\mathcal{L}\left\{y'(t)\right\} = sY(s) - y(0)$$

an *actual* example

- so far, examples just compute $\mathcal{L}\{f(t)\}$ for particular $f(t)$, but they do not show how $\mathcal{L}$ is actually used!
- *example 8*. solve by using $\mathcal{L}$:

$$y' + 5y = t,\ y(0) = 0$$

$$y(t) = \tfrac{1}{25}(e^{-5t} - 1) + \tfrac{t}{5}$$

the old way, to check

- *example 8'*. solve by using Chapter 2 methods:

$$y' + 5y = t, \; y(0) = 0$$

can you always compute $\mathcal{L}$?

- your function $f(t)$ has to be defined on the interval $[0, \infty)$ so you can do the integral $\int_0^\infty e^{-st} f(t)\, dt$
- even then, the function has to *not blow-up too fast*
  - *bad example.* try to compute

$$\mathcal{L}\left\{e^{t^2}\right\} =$$

- the result may not be defined for all s
  - *example.* explain why this result only makes sense for $s > 7$:

$$\mathcal{L}\left\{e^{7t}\right\} = \frac{1}{s-7}$$

can you always compute $\mathcal{L}$? cont.

- *definition.* a function $f(t)$, defined on $[0, \infty)$, is of *exponential order c* if there are constants M and c so that

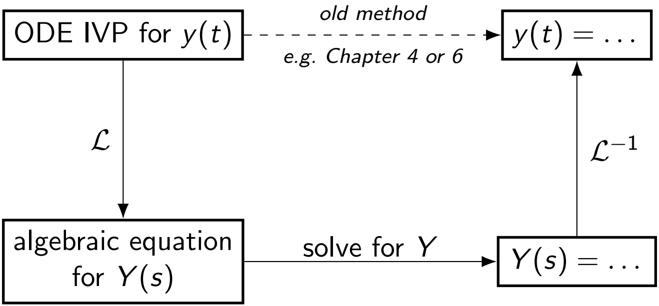
$$|f(t)| \leq Me^{ct}$$

for all t in $[0, \infty)$

Theorem

if $f(t)$ is piecewise continuous on $[0, \infty)$ and of exponential order c then $\mathcal{L}\{f(t)\}$ is defined for $s > c$

the Laplace transform strategy



- example 8 used this strategy
- we get serious about this strategy in §7.2 and §7.3

beating a dead DE . . .

- by then end of this Chapter we will have a *third* good way of solving linear, constant-coefficient DEs:

Chapter 4 use auxiliary equation and undetermined coefficients

Chapter 6 use power series

Chapter 7 use Laplace transform

  - both homogenous and nonhomogeneous
  - all these methods use linearity . . . they are *not* suited to nonlinear DEs
  - only Chapter 6 methods are well-suited to variable coefficients
  - and in Chapter 8 we will get one more method!

expectations

to learn this material, just listening to a lecture is *not* enough

- *read* section 7.1
- find good youtube videos on Laplace transforms?
- do Homework 7.1