6.1 Power series solutions of DEs (and review)

a lecture for MATH F302 Differential Equations

Ed Bueler, Dept. of Mathematics and Statistics, UAF

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for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

we already use power series

- the exponential function is *defined* by an infinite power series:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

  - there are other ways to define it, but via series is the default
  - see characterizations of the exponential function at wikipedia
- generally, a *power series* is an infinite sum of coefficients times powers of x
- *exercise.* from the above series for $y(x) = e^x$, show

$$y' = y \quad \text{and} \quad y(0) = 1$$

a series with unknown coefficients

exercise. find the coefficients in the power series

$$y(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots$$

so that $y(x)$ solves the IVP: $\quad y' + 3y = 0, \quad y(0) = 7$

series solutions of DEs: the basic idea

substitute a series with unknown coefficients into the DE, and thereby find the coefficients

- appropriate initial conditions will yield *one* series solution
- without initial conditions one gets a family of series solutions, the general solution

exercise #37 in §6.1

exercise. find the general solution by using a power series with unknown coefficients:

$$y' = xy$$

review of series

- you already have the main idea ... but reviewing series would be wise
- recall from calculus II:
  1. some familiar series
     - including little tricks for fiddling with familiar series to get other series
  2. how summation notation works
     - including shifting the index of summation
  3. what are the *radius of convergence* and the *interval of convergence*, and how to find them

- I'll do some reviewing in these slides, but ...
- to do *your* review, read the text in section 6.1!

exponential and related series

- we know that for any x,

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

  - $0! = 1$ and $1! = 1$ by definition
  - factorial $n!$ grows faster than b^n for any b ... why? so what?

- split even and odd terms:

$$\cosh x =$$

$$\sinh x =$$

  - $\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}$

- use $e^{i\theta} = \cos\theta + i \sin\theta$:

$$\cos x =$$

$$\sin x =$$

geometric series

- recall:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n$$

  - why?

  - for which x?

related to geometric series

- geometric series for $x \in (-1, 1)$:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots = \sum_{n=0}^{\infty} x^n$$

- substitution gives other series:

$$\frac{1}{1+x^2} =$$

- integration gives other series:

$$\ln(1+x) =$$

$$\arctan(x) =$$

familiar series worth knowing

- . . . I've explained all 8 of these series!

Maclaurin Series	Interval of Convergence
$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$	$(-\infty, \infty)$
$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$(-\infty, \infty)$
$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$(-\infty, \infty)$
$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$	$[-1, 1]$ (2)
$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$	$(-\infty, \infty)$
$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$	$(-\infty, \infty)$
$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$	$(-1, 1]$
$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$	$(-1, 1)$

exercise #14 in §6.1

exercise. Use a familiar series to find the Maclaurin series of the given function. Write your answer in summation notation.

$$f(x) = \frac{x}{1+x^2}$$

base point

- a general power series is

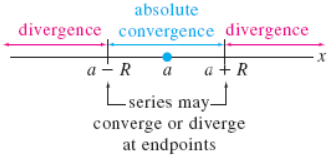
$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

  - a is the *base point*; the series is *centered at a*
  - note that $f(a) = c_0$
  - generally: $\quad c_n = \dfrac{f^{(n)}(a)}{n!}$
- *exercise.* find the power series centered at $a = 5$:

  $f(x) = \sin(2x)$

convergence of power series

- *fact.* for the series there is a value $0 \leq R \leq \infty$ where the series converges if $a - R < x < a + R$ and it diverges if $x < a - R$ or $x > a + R$
  - equivalently "$|x - a| < R$" and "$|x - a| > R$" resp.



- *exercise.* substitute $x = \pm 1$ into

$$\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

do the resulting series converge?

exercise #5 in §6.1

exercise. Find the interval and radius of convergence:

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{10^k}(x-5)^k$$

- using ratio test:

- using geometric series:

exercise #31 in §6.1

exercise. Verify by substitution that the given power series is a solution; use summation notation. Radius of convergence?

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}, \qquad y' + 2xy = 0$$

exercise #31, cont.

expectations

to learn this material, just listening to a lecture is *not* enough

- *read* sections 6.1 and 6.2
- find good youtube videos on power series, for example this one from 3blue1brown:

  www.youtube.com/watch?v=3d6DsjIBzJ4

- do Homework 6.1