5.3 Nonlinear models (with 4.10 material too)

a lecture for MATH F302 Differential Equations

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for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

outline

examples of nonlinear 2nd-order differential equations (DEs):

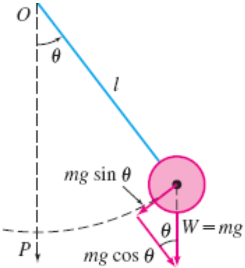
- pendulum (§5.3)
  - using a numerical solver in MATLAB (see §4.10)
- hard and soft springs (§5.3)
- non-constant gravity: from earth to high orbit (§5.3)
- dependent variable missing (§4.10)

nonlinear pendulum

- suppose a pendulum oscillates (swings back and forth) without resistance
- because it oscillates it must be modeled by a 2nd-order linear DE
    - approximately linear for small oscillations
    - for bigger oscillations ($> 20^\circ$?) a nonlinear model is more accurate
- from the diagram:

$$m\ell \frac{d^2\theta}{dt^2} = -mg\sin\theta$$

    - you are not responsible for the derivation
    - but: $s = \ell\theta$ is arclength, so $\ell\frac{d^2\theta}{dt^2}$ is acceleration, and only the tangential force causes motion

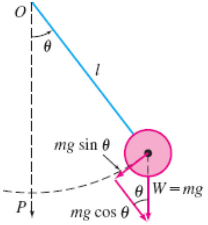


linear small angle model

- divide by $m\ell$ and move term: $\dfrac{d^2\theta}{dt^2} + \dfrac{g}{\ell}\sin\theta = 0$
- if $\omega = \sqrt{\dfrac{g}{\ell}}$ then $\boxed{\dfrac{d^2\theta}{dt^2} + \omega^2 \sin\theta = 0}$ for any angle
- recall $\sin\theta \approx \theta$ for small θ because $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$
- a *small-angle model*:

$$\boxed{\frac{d^2\theta}{dt^2} + \omega^2\theta = 0}$$

  - small-angle solution:
    $\theta(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$



nonlinear versus linearized pendulum

nonlinear: any angles	linearized: small angles
$\theta'' + \omega^2 \sin\theta = 0$	$\theta'' + \omega^2\theta = 0$
solution?	$\theta(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$

- $\omega = \sqrt{g/\ell}$ in both DEs
- we do *not* know how to solve a nonlinear DE like this pendulum
  - the term $\sin\theta$ is not linear: $\sin(a+b) \neq \sin(a) + \sin(b)$

what to do about a nonlinear DE?

- for example, the pendulum DE: $\theta'' + \omega^2 \sin\theta = 0$
- what to do about a nonlinear equation like this?
  - $\theta = e^{rt}$ is not a solution for any r (real or complex)

1. read section 4.10 $\longleftarrow$ *gives advice, not a method*
2. use concept of *energy*
   - makes progress (up-coming worksheet)
   - but we just get a 1st-order DE which we might be unsolveable
3. use *infinite series*
   - makes progress (Chapter 6)
   - but only gives approximations
4. *numerical approximations*
   - Euler's method is just first of many such methods
   - more in Chapter 9
   - requires a specific IVP
   - example next: using an efficient "black box" solver in MATLAB

systems of 1st-order ODEs

need this idea:

a 2nd-order ODE is equivalent to a system of 1st-order ODEs

Example. convert into a 1st-order system:

$$x'' + 5(x')^2 + \sin x = \sqrt{t}$$

Solution. Second derivative $x''(t)$ is merely the derivative of $x'(t)$. So give x' a name:

$$y = x'.$$

Now rewrite $*$ using y:

$$y' + 5y^2 + \sin x = \sqrt{t}.$$

Rearrange above two equations to a system:

$$\begin{aligned} x' &= y \\ y' &= -5y^2 - \sin x + \sqrt{t} \end{aligned}$$

pendulum as a 1st-order system

exercise. convert into a 1st-order system with initial conditions:

$$\theta'' + \omega^2 \sin\theta = 0, \qquad \theta(0) = A, \quad \theta'(0) = B$$

solution.

$$\boxed{\begin{matrix} z_1' = z_2 \\ z_2' = -\omega^2 \sin(z_1) \end{matrix}, \quad \begin{matrix} z_1(0) = A \\ z_2(0) = B \end{matrix}}$$

using black-box solver ode45

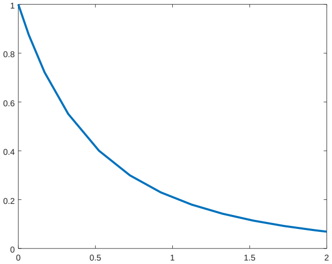
- before we get to numerical solutions of systems, let's do a single 1st-order IVP
- use Matlab or Octave on your own computer or online

example. solve for $y(t)$ on $0 \le t \le 2$, and estimate $y(2)$:

$$y' = -3y + e^{-t}, \quad y(0) = 1$$

solution. the DE is $y' = f(t, y)$ so

```
>> f = @(t,y) -3*y + exp(-t);
>> [tt,yy] = ode45(f,[0,2],1);
>> plot(tt,yy)
>> yy(end)
ans =  0.068908
```



only 12 steps, but accurate

- the `ode45` black-box is quite accurate
- *exercise.* solve *by hand* for the exact value $y(2)$:

$$y' = -3y + e^{-t}, \quad y(0) = 1$$

solution.

- compare to `y(end)=y(13)` on previous slides:

```
>> 0.5*(exp(-2)+exp(-6))
ans =  0.068907
```

- Euler would need 10^5 or 10^6 steps for this accuracy

calling ode45

- from the MATLAB documentation page on ode45:

  `[t,y] = ode45(odefun,tspan,y0)`,

  where `tspan = [t0 tf]`, integrates the system of differential equations $y' = f(t, y)$ from `t0` to `tf` with initial conditions `y0`. Each row in the solution array `y` corresponds to a value returned in column vector `t`.

- see the above MATLAB page for examples of functions $f(t, y)$ for the `odefun` argument
- note further fine print about the `tspan` argument:
  - If tspan has two elements `[t0 tf]` then the solver returns the solution evaluated at internal integration steps in the interval.
  - If tspan has more than two elements `[t0,t1,t2,...,tf]` then the solver returns the solution evaluated at the given points.

ode45 for pendulum

example. let $\omega = \sqrt{7}$. solve for $\theta(t)$ on the interval $t \in [0, 20]$:

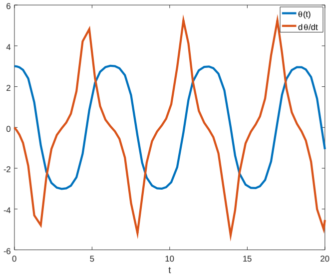
$$\theta'' + \omega^2 \sin\theta = 0, \qquad \theta(0) = 3, \quad \theta'(0) = 0$$

solution. $z_1 = \theta$ and $\omega^2 = 7$ so

$$\begin{aligned} z_1' &= z_2 & \qquad z_1(0) &= 3 \\ z_2' &= -7\sin(z_1) & \qquad z_2(0) &= 0 \end{aligned}$$

This is $z' = f(t, z)$ so:

```
>> f = @(t,z) [z(2); -7*sin(z(1))];
>> [tt,zz] = ode45(f,[0,20],[3;0]);
>> plot(tt,zz)
>> xlabel t
>> legend('\theta(t)','d\theta/dt')
```

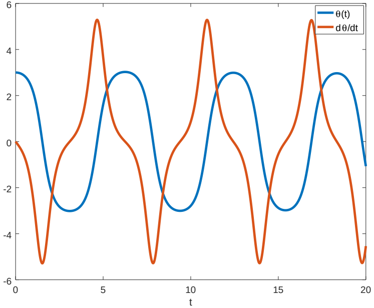


pendulum: better and movier

- the solution is more accurate than it looks!
- for better appearance, generate more points (below):

```
>> [tt,zz] = ode45(f,[0:.01:20],[3;0]);
>> plot(tt,zz),  xlabel t
```

- one can also make a movie
  - see `pendmovie.m` at the public Codes tab



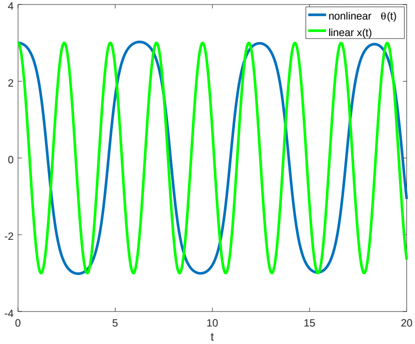
back to linear mass-spring

example. solve for $x(t)$ on the interval $t \in [0, 20]$:

$$x'' + 7x = 0, \qquad x(0) = 3, \quad x'(0) = 0$$

exact solution.

$$x(t) = 3\cos(\sqrt{7}t)$$



continuing previous code:

```
>> plot(tt,zz(:,1),'b',tt,3*cos(sqrt(7)*tt),'g')
>> xlabel t
>> legend('nonlinear \theta(t)','linear x(t)')
```

linear mass-spring: exact vs. numerical

- this is a good case on which to check accuracy
- *example.* find $x(20)$:

$$x'' + 7x = 0, \qquad x(0) = 3, \quad x'(0) = 0$$

exact solution. $x(20) = 3\cos(\sqrt{7}(20)) = -2.6441$

numerical solution. $z_1 = x$ and $z_2 = x'$ so

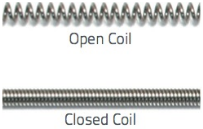
$$\begin{aligned} z_1' &= z_2 & z_1(0) &= 3 \\ z_2' &= -7z_1 & z_2(0) &= 0 \end{aligned}$$

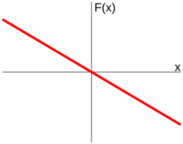
```
>> fl = @(t,z) [z(2); -7*z(1)];
>> [ttl,zzl] = ode45(fl,[0:.01:20],[3;0]);
>> zzl(end,1)
ans = -2.6492
```

- what about plots of the exact and numerical solutions?
  - you won't see difference: $x(t) = 3\cos(\sqrt{7}t)$ versus `zzl(:,1)`

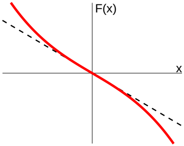
nonlinear springs

- springs are usually well-modeled by Hooke's law $F(x) = -kx$ for small displacements x from the equilibrium position
- . . . *but* when they are over-extended, or closed coil, etc. then they need different models $mx'' = F(x)$

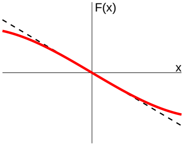




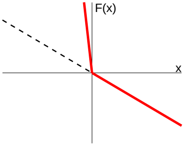
linear



hard



soft



closed

exercise #9: (numerical) nonlinear spring

- so $F(x) = -x - x^3$ is a hard spring model
- suppose we also have damping (thus $x(t) \to 0$ as $t \to \infty$)

exercise #9 in §5.3: numerically solve

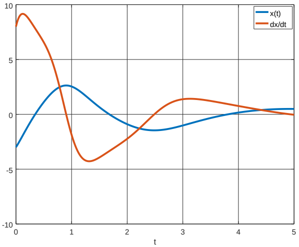
$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x + x^3 = 0, \quad x(0) = -3, x'(0) = 8$$

solution: write as system using $x = z_1$, $x' = z_2$:

$$\begin{aligned} z_1' &= z_2 & z_1(0) &= -3 \\ z_2' &= -z_2 - z_1 - z_1^3 & z_2(0) &= 8 \end{aligned}$$

and use ode45:

```
>> f = @(t,z) [z(2); -z(2)-z(1)-z(1)^3];
>> [tt,zz] = ode45(f,[0:.01:5],[-3;8]);
>> plot(tt,zz), xlabel t, grid on
>> legend('x(t)','dx/dt')
```



bullet to geosynchronous orbit

example. We want to use a bullet weighting 100 grams to destroy a satellite in geosynchronous (geostationary) orbit, approximately 36000 km. What velocity is needed if we ignore air drag?

solution. Constant gravity g will *not* do. The gravity decreases as the bullet rises. §5.3 states Newton's law of gravitation:

$$my'' = -k\frac{Mm}{y^2} \text{ where } m = \text{(bullet mass)}, \; M = \text{(earth mass)}$$

After simplification (see text), and with initial conditions, this is

$$y'' = -g\frac{R^2}{y^2}, \qquad y(0) = R, \quad y'(0) = V$$

We take $R = 6.4 \times 10^6$ m =(radius of earth) and $g = 9.8$. (*Note bullet mass does not matter. Earth's mass is built into* g.)

The Question: Find V so that the maximum of $y(t)$ solving the above IVP is 3.6×10^7 m.

bullet to geosynchronous orbit 2

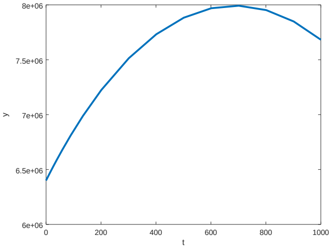
question: Find V so $\max y(t) = 3.6 \times 10^7$, given

$$y'' = -g\frac{R^2}{y^2}, \qquad y(0) = R, \quad y'(0) = V$$

and $R = 6.4 \times 10^6$ m =(radius of earth) and $g = 9.8$
solution?: as system with $y = z_1$, $y' = z_2$ and $C = gR^2$:

$$z_1' = z_2 \qquad\qquad z_1(0) = R$$
$$z_2' = -Cz_1^{-2} \qquad\qquad z_2(0) = V$$

```
>> g = 9.8;  R = 6.4e6;  C = g*R^2;
>> f = @(t,z) [z(2); -C/z(1)^2];
>> V = 5000;
>> [tt,zz] = ode45(f,[0,1000],[R;V]);
>> plot(tt,zz(:,1))
>> xlabel t,  ylabel y
>> max(zz(:,1))
ans =    7.9924e+06
```



bullet to geosynchronous orbit 3

- trial and error needed!
- I finished with:

```
>> V = 10157; [tt,zz] = ode45(f,[0,20000],[R;V]);
>> [max(zz(:,1)) zz(end,1)]
ans =
  3.60120e+07  2.36604e+07
```

a bit of hard-earned extra credit for any of these:

1. energy methods allow you to solve the above problem *by hand*; see upcoming worksheet on how to do it,
2. but on the other hand one can add air drag by a reasonable model and use the *same* numerical method from MATLAB; do so
3. given air drag from 2, will the bullet survive the heating? (ablative ceramic-coated tungsten bullet?)
   - this will need another DE coupled to the first

how the black box works

- how does the black box `ode45` work?
    - good question!
- *basically*: it is just a fancier form of Euler's method
- *more thoroughly*:
    - it uses a pair of Runge-Kutta methods
    - ...so it can adaptively choose its step size
    - see the MATLAB reference page for `ode45`
    - covered in Chapter 9

dependent variable missing

- there are by-hand solvable nonlinear 2nd-order DEs:

DE	technique	first integral
$y'' = f(t, y, y')$	*too general*	
$y'' = f(t)$	just antidifferentiate	$y' = F(t) + c$ where $F(t) = \int f(t)\,dt$
$y'' = f(y)$	compute energy *[worksheet]*	$\frac{1}{2}(y')^2 + P(y) = c$ where $P(z) = -\int f(z)\,dz$
$y'' = f(y')$	substitute $u = y'$ *[§4.10]*	$Q(y') = t + c$ where $Q(u) = \int \frac{du}{f(u)}$

- last category called "dependent variable y is missing" (§4.10)
- you can often solve by the substitution $u = y'$
  - this can sometimes work for $y'' = f(t, y')$ too

exercise #6 in §4.10

exercise. find the general solution:

$$e^{-t}y'' = (y')^2$$

expectations

to learn this material, just listening to a lecture is *not* enough

- *read* section 4.10 in the textbook
  - skip the "Use of Taylor series" material ... we'll get to it later
- *read* section 5.3 in the textbook
  - you can safely skip the material on "Telephone wires" (boundary value problems are not covered in Math 302)
- take the whole thing seriously by going and finding some good youtube videos etc. on ODE simulations
- do Homework 5.3