5.1 Linear mass-spring models

a lecture for MATH F302 Differential Equations

Ed Bueler, Dept. of Mathematics and Statistics, UAF

Fall 2023

for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

a good reason

- in Chapter 4 we solved 2nd-order linear DEs

$$ay'' + by' + cy \overset{*}{=} g(t)$$

- a good reason is that

  anything that smoothly oscillates has $*$ for a model

  1. a mass suspended on a spring oscillates up and down
  2. the current in an electrical circuit flows back-and-forth
  3. a pendulum swings back and forth
  4. the earth moves up and down in an earthquake
  5. magnetic field in a radio wave oscillates
  6. a drum-head vibrates
  7. a photon is

- 5.1 and 5.3 slides cover ❶ – ❸

1st-order linear: no oscillation

- *background assumption*: laws of nature are autonomous
- why is 2nd-order needed for oscillation?
- 1st-order linear autonomous DEs cannot generate oscillation

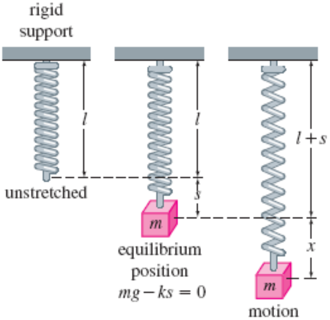
$$y' = ay + b$$
$$\int \frac{dy}{ay+b} = \int dt$$
$$\frac{1}{a} \ln |ay+b| = t + c$$
$$y(t) = \frac{1}{a}\left(Ce^{at} - b\right)$$

  - solutions are *always* growing/decaying exponentials
- 1st-order *non*linear DEs would be nearly-linear for small solutions
- summary: we expect oscillation models are 2nd-order
  - we know examples: $y'' + y = 0 \iff y = c_1 \cos t + c_2 \sin t$

mass-spring model: the setup

a specific set-up so that the equations are clear:

- hang spring from rigid support
  - length ℓ and spring constant k
- choose mass m and hook to the spring
- it stretchs distance s down to equilibrium position
- mark length scale:
  - $x = 0$ is equilibrium position
  - positive x is downward
- x is the displacement from additional stretch of the spring, i.e. *downward displacement of the mass from its equilibrium position*



Newton's law

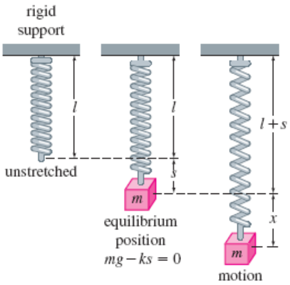
- Newton's second law is $ma = F$
- for our first mass-spring model:

$$m\frac{d^2x}{dt^2} = mg - k(x+s)$$

  - but $mg = ks$ so

$$\boxed{m\frac{d^2x}{dt^2} = -kx}$$

- "Hooke's law" $F_{spring} = -kx$ is a *model* for how springs work
  - not a bad model for small motions
  - improved model in 5.3
- in practice:
  k is <u>determined</u> from $mg = ks$



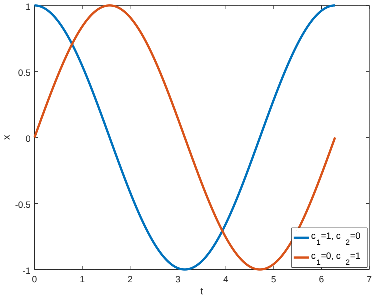
(undamped) mass-spring solution

- from last slide: $m\frac{d^2x}{dt^2} + kx = 0$
- constant coefficient: substitute $x(t) = e^{rt}$ and get

$$mr^2 + k = 0 \qquad \Longleftrightarrow \qquad r = \pm\sqrt{\frac{k}{m}}i = \pm\omega i$$

- $\omega = \sqrt{\frac{k}{m}}$
- general solution:

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$



the meaning of ω

- general solution: $x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$
- suppose t is measured in seconds
- then $\omega = \sqrt{\frac{k}{m}}$ is *frequency of oscillation* in radians per second
    - units are correct because ωt must be in radians
- time $T = \frac{2\pi}{\omega}$ is *period of oscillation*
    - equation $\omega T = 2\pi$ gives the smallest $T > 0$ so that

$$\cos(\omega T) = \cos(0) \quad \text{and} \quad \sin(\omega T) = \sin(0)$$

    - ...general solution has period T

§5.1 exer. #3: "free undamped motion"

3. A mass weighing 24 pounds, attached to the end of a spring, stretches it 4 inches. Initially the mass is released from rest from a point 3 inches above the equilibrium position. Find the equation of motion.

mass/weight English unit stupidity

- "kilograms" is the SI unit for *mass* m
  - $g = 9.8\,\mathrm{m/s^2}$ is acceleration of gravity
  - mg is a force in newtons $N = \mathrm{kg\,m/s^2}$
- "pounds" is a unit for *force* mg
  - it is a *weight* not a mass
- "slugs" are a unit for *mass* m
  - old English system . . .
  - and you need: $g = 32\,\mathrm{ft/s^2}$

amplitude and phase of $x(t)$

- for any c_1, c_2, this formula is a wave or oscillation:

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

- what is its amplitude?
  - only an easy question if either $c_1 = 0$ or $c_2 = 0$

Problem: find *amplitude* A and *phase angle* ϕ so that

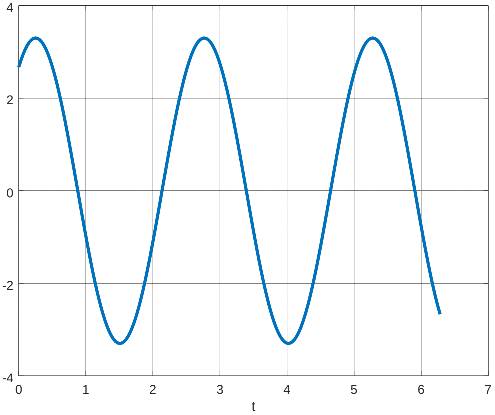
$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t = A \sin(\omega t + \phi)$$

Solution: use $\sin(a+b) = \sin a \cos b + \cos a \sin b$ so

$$\begin{aligned} A \sin(\omega t + \phi) &= A \sin(\omega t) \cos \phi + A \cos(\omega t) \sin \phi \\ \implies \qquad & c_1 = A \sin \phi, \ c_2 = A \cos \phi \\ \implies \qquad & A = \sqrt{c_1^2 + c_2^2}, \quad \tan \phi = \frac{c_1}{c_2} \end{aligned}$$

illustration

- *example:* graph $x(t) = A\sin(\omega t + \phi)$ for frequency $\omega = 2.7$, amplitude $A = 3.3$, and phase angle $\phi = 0.3\pi$
  - period $T = 2\pi/\omega = 2.51$
  - $x(t) = 2.67\cos(\omega t) + 1.94\sin(\omega t)$



exercise #6 in §5.1

- another "free undamped motion" exercise

  6. A force of 400 newtons stretches a spring 2 meters. A mass of 50 kilograms is attached to the end of the spring and is initially released from the equilibrium position with an upward velocity of 10 m/s. Find the motion $x(t)$.

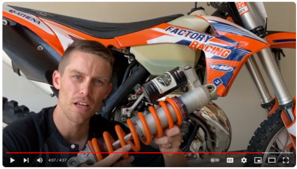
damped mass-spring model

- actual mass-springs don't oscillate forever
- friction or drag is called "damping"
  - simple case: mass is surrounded by water or other fluid
- *model*: damping is proportional to velocity

$$F_{\text{damping}} = -\beta v = -\beta \frac{dx}{dt}$$

  - $\beta > 0$ so damping force opposes motion
  - same model as drag force for projectiles in sections 1.3, 3.1
- Newton's 2nd law again:

$$\boxed{m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt}} \qquad \text{or} \qquad mx'' = -kx - \beta x'$$



youtu.be/lPg695IXbPo

damped solution method

- recall *undamped* mass-spring model with $\omega = \sqrt{\frac{k}{m}}$:

$$mx'' = -kx \quad \iff \quad x'' + \omega^2 x = 0$$

- new damped mass-spring model:

$$mx'' = -kx - \beta x' \quad \iff \quad x'' + 2\lambda x' + \omega^2 x = 0$$

  - $\lambda = \dfrac{\beta}{2m}$
  - auxiliary equation from $x(t) = e^{rt}$:

$$r^2 + 2\lambda r + \omega^2 = 0$$

  - has roots:

$$r = \frac{-2\lambda \pm \sqrt{4\lambda^2 - 4\omega^2}}{2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2} = r_1, r_2$$

  - are r_1, r_2 distinct? real? complex?

exercise #27 in §5.1

- "free damped motion" exercise

  27. A 1 kilogram mass is attached to a spring whose constant is 16 *N/m. The entire system is submerged in a liquid that imparts a damping force numerically equal to 10 times the instantaneous velocity. Determine the equations of motion if the mass is initially released from rest from a point 1 meter below the equilibrium position.*

slight variation comes out different

A 1 kilogram mass is attached to a spring whose constant is 16 N/m. The entire system is submerged in a liquid that imparts a damping force numerically equal to 6 times the instantaneous velocity. Determine the equations of motion if the mass is initially released from rest from a point 1 meter below the equilibrium position.

damping cases

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

- *undamped* if $\lambda = 0$:

$$x(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$$

- *overdamped* if $\lambda^2 - \omega^2 > 0$: $\boxed{r_1, r_2 = -\lambda \pm \sqrt{\lambda^2 - \omega^2}}$

$$x(t) = e^{-\lambda t}\left(c_1 e^{\sqrt{\lambda^2-\omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}\,t}\right)$$

- *critically damped* if $\lambda^2 - \omega^2 = 0$: $\boxed{r_1 = r_2 = -\lambda}$

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

- *underdamped* if $\lambda^2 - \omega^2 < 0$: $\boxed{r_1, r_2 = -\lambda \pm \sqrt{\omega^2 - \lambda^2}\,i}$

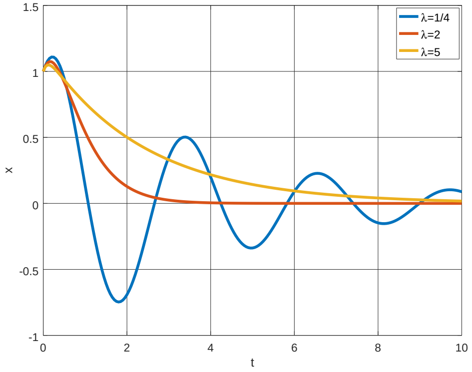
$$x(t) = e^{-\lambda t}\left(c_1\cos(\sqrt{\omega^2-\lambda^2}\,t) + c_2\sin(\sqrt{\omega^2-\lambda^2}\,t)\right)$$

damping cases pictured

- consider $m = 1,\ k = 4$
- $\omega = \sqrt{\frac{k}{m}} = 2$:

$$\frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + 4x = 0$$

- with initial values $x(0) = 1, x'(0) = 1$
- picture cases $\lambda = 1/4, 2, 5$
    - recall $\lambda = \frac{\beta}{2m}$
    - so $\beta = 1/2, 4, 10$



a plotting code: `massspringplot.m`

```
function massspringplot(m,beta,k,x0,v0,T)
% MASSSPRINGPLOT  Make a plot on 0 < t < T of solution to
%     m x'' + beta x' + k x = 0
% with initial conditions x(0) = x0, x'(0) = v0.

omega = sqrt(k/m);  lambda = beta/(2*m);
D = lambda^2 - omega^2;
t = 0:T/200:T;   % 200 points enough for smooth graph
if D > 0
    fprintf('overdamped\n')
    Z = sqrt(D);   c = [1, 1; -lambda+Z, -lambda-Z] \ [x0; v0];
    x = exp(-lambda*t) .* (c(1) * exp(Z*t) + c(2) * exp(-Z*t));
elseif D == 0
    fprintf('critically damped\n')
    c = [x0; v0 + lambda * x0];
    x = exp(-lambda*t) .* (c(1) + c(2) * t);
else  % D < 0
    fprintf('underdamped\n')
    W = sqrt(-D);   c = [x0; (v0 + lambda * x0) / W];
    x = exp(-lambda*t) .* (c(1) * cos(W*t) + c(2) * sin(W*t));
end
plot(t,x),  grid on,  xlabel('t'),  ylabel('x')
```

example: solve the IVP

$$mx'' = -kx - \beta x', \qquad x(0) = x_0,\, x'(0) = v_0$$

in the critically-damped case

forced

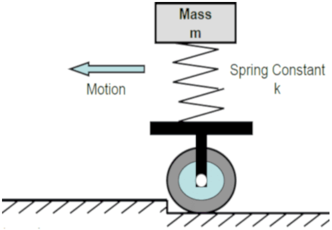
- the nonhomogeneous version is called a *driven*, damped mass-spring where force $f(t)$ is applied to the mass:

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} + f(t)$$

- equivalently, after dividing by m:

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

- a version of this model is a damped mass-spring formed by your car
    - force is applied to the support and your car is the mass



mass-spring DEs

	Newton's law: $ma = F$	ω form
undamped	$m\frac{d^2x}{dt^2} = -kx$	$\frac{d^2x}{dt^2} + \omega^2 x = 0$
damped	$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt}$	$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$
damped and driven	$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} + f(t)$	$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$

notes:

- $\omega = \sqrt{k/m}$, $\lambda = \beta/(2m)$, $F(t) = f(t)/m$
- with driving force $f(t)$ the problem is *nonhomogeneous*
- you would solve the damped and driven problems by undetermined coefficients to find a particular solution (section 4.4)

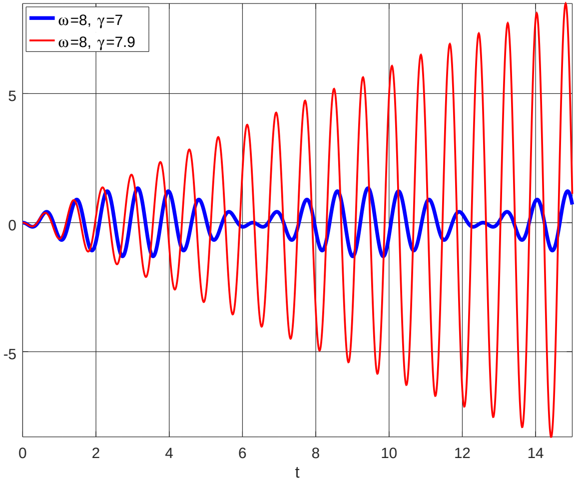
exercise #43 in §5.1

Solve the IVP

$$\frac{d^2x}{dt^2} + \omega^2 x = F_0 \cos \gamma t, \qquad x(0) = 0, \quad x'(0) = 0$$

and compute $\lim_{\gamma \to \omega} x(t)$

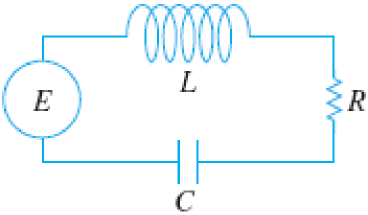
exercise #43 pictured



- idea: resonance can occur in driven mass-spring systems

RLC circuit

- consider the electrical circuit:



- has electical source ($E = E(t)$), an inductor (L), a resistor (R), and a capacitor (C)
- a differential equation for the *charge* q is

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

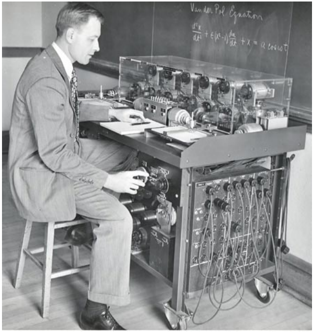
- because $dq/dt = I$, a differential equation for the *current* I is

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = E'(t)$$

circuit analogy

mass-spring	electical circuit
mass m	inductance L
drag β	resistance R
spring constant k	inverse of capacitance $1/C$
applied driving force $f(t)$	applied voltage source $E(t)$
$mx'' + \beta x' + kx = f(t)$	$Lq'' + Rq' + \frac{1}{C} q = E(t)$

- this is how radios are understood
    - *tuning a radio* means choosing the capacitance C to cause resonance at the frequency you want to hear from the input $E(t)$ from the antenna
- based on this idea there were *analog computers* which used a configurable electical circuit to model mechanical motions



expectations

to learn this material, just listening to a lecture is *not* enough

- *read* section 5.1 in the textbook
  - material on "double spring systems" (p. 201) can be skipped
  - while I discussed electrical circuits in these slides, I will not ask about it on quizzes or exams
- do Homework 5.1