4.4 Nonhomogeneous equations: method of undetermined coefficients

a lecture for MATH F302 Differential Equations

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for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

general solutions to nonhomogeneous DEs

- for an *n*th-order, linear, and nonhomogeneous DE

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y \overset{*}{=} g(x)$$

- . . . the general solution is a sum of the general solution of the associated *homogeneous* equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$$

  plus one *particular solution* $y_p(x)$ of $*$
  - the general solution of the homogeneous equation is called the *complementary function* $y_c(x)$

- main structure: $y(x) = y_c(x) + y_p(x)$ solves $*$

- *example 1*: find the general solution:

$$y'' + 4y = e^{-x}$$

example 1, cont.

- verify that $y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5}e^{-x}$ solves

$$y'' + 4y = e^{-x}$$

example 1, cont.[2]

- solve the initial value problem:

$$y'' + 4y = e^{-x}, \qquad y(0) = -1, \quad y'(0) = 1$$

example 2

- the idea of "undetermined coefficients" is to try $y_p(x)$ which has the same general form as the nonhomogeneity $g(x)$
- *example 2 (≈ #5 in 4.4)*: find the general solution:

$$y'' + 4y' + 4y = x^2 - 2x$$

example 3

- *example 3 (#8 in 4.4)*: find the general solution:

$$4y'' - 4y' - 3y = \cos 2x$$

trial forms for the particular solution

- we need some guidance on how to guess!
- in words:
  For y_p try a linear combination of all linearly-independent functions generated by repeated differentiation of $g(x)$.
- as a table:

TABLE 4.4.1 Trial Particular Solutions

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A \cos 4x + B \sin 4x$
6. $\cos 4x$	$A \cos 4x + B \sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x} \sin 4x$	$Ae^{3x} \cos 4x + Be^{3x} \sin 4x$
11. $5x^2 \sin 4x$	$(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$
12. $xe^{3x} \cos 4x$	$(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x$

example 4 *shows we still have issues!*

- *example 4 (≈ #13 in 4.4)*: find the general solution:

$$y'' + 9y = 2\cos 3x$$

guidance on the hard case

- the problematic case happens when our guess for y_p "accidently" contain terms which also appear in y_c
  - because the left side then annihilates those terms
  - ...which blocks us from determining y_p

- guidance in words:
  If the trial form of y_p contains terms that duplicate terms in y_c then multiply the trial form by x^n where n is the smallest power that eliminates the duplication.

example 5

- *example 5 (#29 in 4.4)*: solve the initial value problem:

$$5y'' + y' = -6x, \qquad y(0) = 0, \quad y'(0) = -10$$

example 6

- *example 6 (#32 in 4.4)*: solve the initial value problem:

$$y'' - y = \cosh x, \qquad y(0) = 2, \quad y'(0) = 12$$

example 6, cont.

- the last slide had an impressive calculation, so we should ...
- verify that $y(x) = 7e^x - 5e^{-x} + \frac{1}{4}xe^x - \frac{1}{4}xe^{-x}$ solves $y'' - y = \cosh x,\ y(0) = 2, y'(0) = 12$

clearly

- clearly you need to practice examples, not just me

what we are skipping next

we are skipping the following sections:

- §4.5 *Undetermined Coefficients—Annihilator Approach*
  a more abstract view of undetermined coefficients ... but no more powerful than our superposition method
- §4.6 *Variation of parameters*
  a general approach to nonhomogeneous linear equations but one may not be able to compute the integrals you get
  - it is somewhat like reduction of order in §4.2
- §4.7 *Cauchy-Euler equations*
  another class of homogeneous differential equations which can be solved via an auxiliary equation
- §4.8 *Green's Functions*
  mostly relevant to boundary value problems (not in Math 302)
- §4.9 *Solving Systems of Linear DEs by Elimination*
  a way of solving systems ... which are important ... but done generally and powerfully in chapter 8

expectations

to learn this material, just listening to a lecture is *not* enough

- *read* section 4.4 in the textbook
- do Homework 4.4
- find YouTube worked examples; search “nonhomogeneous differential equations” and “method of undetermined coefficients”