4.3 Homogeneous linear equations with constant coefficients

a lecture for MATH F302 Differential Equations

Ed Bueler, Dept. of Mathematics and Statistics, UAF

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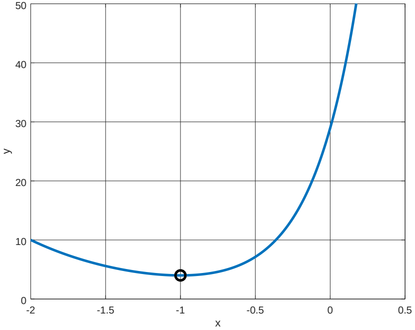
for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

linear, homogeneous, constant-coefficient

- recall from §4.1 slides that linear DEs which are homogeneous and constant-coefficient always have exponential solutions
  - *fact*: you can always find at least one solution $y = e^{mx}$
  - but each of the underlined words is important to this fact
- *example 1*: solve the ODE IVP

$$y'' - 2y' - 4y = 0, \quad y(-1) = 4,\ y'(-1) = 0$$

example 1, finished



example 1: how I did it

- here is how I solved for the constants and made the figure using Matlab:

```
w = 1-sqrt(5);  z = 1+sqrt(5);
A = [exp(-w), exp(-z); w*exp(-w), z*exp(-z)];
b = [4; 0];
c = A \ b           % get: c(1)=0.8409, c(2)=28.119

x = -2:.01:1;
y = c(1) * exp(w*x) + c(2) * exp(z*x);
plot(x,y),  grid on,  xlabel x,  ylabel y
axis([-2 0.5 0 50])
hold on,  plot(-1,4,'ko','markersize',12),  hold off
```

- I am committed to helping you use a computer for math!

example 2

- *example 2*: find the general solution of the ODE

$$y'' + y = 0$$

Euler's helpful identity

- Euler recognized the connection between imaginary numbers and trig functions:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

- *exercise*: Explain *Euler's identity* above using the Taylor series of $e^x, \cos x, \sin x$ at basepoint $x_0 = 0$. Also draw a picture.

example 3

- from Euler's identity we also know

  $$e^{a+ib} = e^a(\cos b + i \sin b)$$

- *example 3*: find the general solution of the ODE

$$y'' - 4y' + 5y = 0$$

the major facts of §4.3

for constant-coefficient and homogeneous linear ODEs

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

- substitution of $y = e^{mx}$ yields (polynomial) *auxiliary equation*

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

- any polynomial eqn. has at least one *complex* root (solution)
  - auxiliary eqn. has at least 1 and at most n distinct roots
  - some roots may be repeated
- there is a recipe (next slide!) which generates a fundamental set of n real solutions and a general solution to the ODE:

$$y_1(x), \dots, y_n(x) \quad \Longrightarrow \quad y(x) = c_1 y_1(x) + \cdots + c_n y_n(x)$$

main recipe of §4.3

find all roots of the auxiliary equation

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

and then build a fundamental solution set this way:

case I: if m is a real root then

$$e^{mx} \text{ is in the set}$$

case II: if m is a real root which is repeated k times then

$$e^{mx},\ xe^{mx},\ \dots,\ x^{k-1}e^{mx} \text{ are in the set}$$

case III: if $m = a \pm ib$ is a complex root then

$$e^{ax}\cos(bx),\ e^{ax}\sin(bx) \text{ are in the set}$$

exercise 5 in §4.3

- *exercise 5*: find the general solution of the second-order DE

$$y'' + 8y' + 16y = 0$$

exercise 23 in §4.3

- *exercise 23*: find the general solution of the higher-order DE

$$y^{(4)} + y''' + y'' = 0$$

exercise 55 in §4.3

- *exercise 55*: find a constant-coefficient, homogeneous linear DE whose general solution is

$$y(x) = c_1 e^{-x} \cos x + c_2 e^{-x} \sin x$$

like exercise 69 in §4.3

- *like exercise 69*: solve the ODE IVP

$$2y^{(4)} + 13y''' + 21y'' + 2y' - 8y = 0$$
$$y(0) = -2,\ y'(0) = 6,\ y''(0) = 3,\ y'''(0) = \tfrac{1}{2}$$

hint. you may use a computer algebra system (CAS)

exercise 69: how to do it

```
>> m = roots([2,13,21,2,-8])'
m =
          -4            -2            -1           0.5
>> A = [1 1 1 1; m; m.^2; m.^3]
A =
           1             1             1             1
          -4            -2            -1           0.5
          16             4             1          0.25
         -64            -8            -1         0.125
>> b = [-2 6 3 0.5]';
>> c = A \ b
c =
    -0.48148
         5.4
     -12.222
      5.3037
```

conclusion: a computer is very effective . . . if you know where you are going

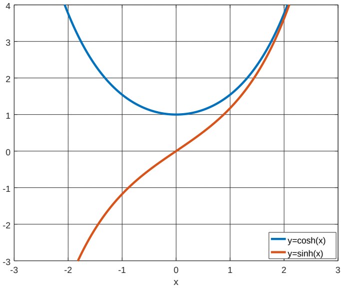
hyperbolic functions

- Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$, for complex exponentials, has an analog for real exponentials

- by definition:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\sinh x = \frac{e^x - e^{-x}}{2}$$

  - the even and odd parts of the exponential, resp.
  - called *hyperbolic* functions



- it is easy to see that
  - $e^x = \cosh x + \sinh x$
  - $(\cosh x)' = \sinh x, \quad (\sinh x)' = \cosh x$
  - $y = c_1 \cosh x + c_2 \sinh x$ is a general solution to $y'' - y = 0$

some nice cases

- the following general solutions can all be computed by substituting $y = e^{mx}$, and getting the auxiliary equation, etc.
- ...but it is good to *quickly* apply these special cases:

$$\begin{array}{rcc} & \text{has general solution} & \\ y' = ky & \longrightarrow & y = Ae^{kx} \\ y'' + k^2 y = 0 & \longrightarrow & y = c_1 \cos(kx) + c_2 \sin(kx) \\ y'' - k^2 y = 0 & \longrightarrow & \begin{bmatrix} y = c_1 e^{kx} + c_2 e^{-kx} \\ \text{or} \\ y = b_1 \cosh(kx) + b_2 \sinh(kx) \end{bmatrix} \\ y'' = 0 & \longrightarrow & y = c_1 + c_2 x \end{array}$$

expectations

to learn this material, just listening to a lecture is *not* enough

- *read* section 4.3 in the textbook
- find YouTube videos on "second order differential equations"
  ... then look for the constant-coefficient case
- for §4.3 you at least need to know these terms:

  homogeneous
  linearly (in)dependent
  Wronskian
  fundamental set of solutions
  linear combination
  general solution

- when we go back to §4.2: why does the repeated-roots case generate additional linearly-independent solutions via extra factors of "x"?