4.2 Reduction of order

a lecture for MATH F302 Differential Equations

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Fall 2023

for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

a 2nd-order example

- for this example, §4.3 methods *do not* work
  ...but we can find the solution anyway
- *example 1*: find the general solution: $xy'' + y' = 0$

the §4.3 rule needing explanation

- the next example is one we *do* know how to solve
  ... but via a §4.3 rule for which I owe you justification
- *example 2*: find the general solution: $y'' - 6y' + 9y = 0$

reducing the order 1: first illustration

- *reduction of order* is a technique:
  - substitute $y(x) = u(x)y_1(x)$
  - derive a DE for u which has no zeroth-order term
  - solve a first-order equation for $w = u'$
- key understanding: the purpose is to find another linearly-independent solution given you have $y_1(x)$
- *example 2 again*: $y_1(x) = e^{3x}$ is known; find another

  $$y'' - 6y' + 9y = 0$$

reducing the order 2: general case

suppose $y_1(x)$ is a solution to 2nd-order homogeneous DE

$$y'' + P(x)y' + Q(x)y \stackrel{*}{=} 0,$$

and we seek another solution of the form $y(x) = u(x)y_1(x)$:

- compute $y' = u'y_1 + uy_1'$ and $y'' = u''y_1 + 2u'y_1' + uy_1''$

  check it: $y'' =$

- substitute into $*$:

$$(u''y_1 + 2u'y_1' + uy_1'') + P(u'y_1 + uy_1') + Quy_1 = 0$$

- group by derivatives on u:

$$y_1u'' + (2y_1' + Py_1)u' + (y_1'' + Py_1' + Qy_1)u = 0$$

- term in green is zero (*why?*) so u solves

$$y_1u'' + (2y_1' + Py_1)u' = 0$$

reducing the order 3: a first-order equation

- we are seeking a solution of the form $y = uy_1$, and u solves

$$y_1 u'' + (2y_1' + Py_1)u' = 0$$

- there is no zeroth-order term so we can solve it
- the equation is first-order and separable for $w = u'$:

$$y_1 w' + (2y_1' + Py_1)w = 0$$

$$\frac{dw}{dx} = -\frac{(2y_1' + Py_1)w}{y_1}$$

$$\frac{dw}{w} = -\left(2\frac{y_1'}{y_1} + P\right)\, dx$$

$$\int \frac{dw}{w} = -2\int \frac{y_1'(x)}{y_1(x)}\, dx - \int P(x)\, dx$$

reducing the order 4: the second solution

- continuing:

$$\int \frac{dw}{w} = -2 \int \frac{y_1'(x)}{y_1(x)}\,dx - \int P(x)\,dx$$

$$\ln|w(x)| = -2\ln|y_1(x)| - \int P(x)\,dx + C$$

$$w(x) = c_1 \frac{e^{-\int P(x)\,dx}}{y_1(x)^2}$$

- recall $u' = w$; thus integrating again gives

$$u(x) = c_1 \int \frac{e^{-\int P(x)\,dx}}{y_1(x)^2}\,dx + c_2$$

- the second solution is the new part of $y = uy_1$:

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1(x)^2}\,dx$$

example 4

- in this example we tell a complete story: we *guess* a first solution and then derive a second one by reduction of order
- *example 4*: find the general solution (for $x > 0$)

$$x^2 y'' + 5xy' + 4y = 0$$

example 4, finished

$$y(x) = c_1 x^{-2} + c_2 x^{-2} \ln x$$

expectations

to learn this material, just listening to a lecture is *not* enough

- *read* section 4.2 in the textbook
- note example 4 is a *Cauchy-Euler* type of differential equation
  - covered in §4.7 . . . which we will otherwise skip
- to do reduction of order on a quiz or exam you have a choice
- do you

  1. memorize $\boxed{y_2(x) = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1(x)^2}\,dx}$?
  2. or substitute $y(x) = u(x)y_1(x)$ and see how it comes out?