4.1 Higher-order linear equations: first examples and preliminaries

a lecture for MATH F302 Differential Equations

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for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

outline

plan for these slides

- a bit of review of first-order linear equations (§2.3)
- a first look at how to solve constant-coefficient, second-order linear equations (from §4.3)
- a whole bunch of new language for higher-order linear equations
    - basically, §4.1 is a lot of new words

first-order linear DEs: a review

- recall first-order linear DEs:

$$a_1(x)y' + a_0(x)y = g(x)$$

- one may divide by the leading coefficient:

$$y' + P(x)y = f(x)$$

  - this requires leading coefficient $a_1(x)$ to *not* to be zero on the interval where we are solving

- special case 1 (easiest to solve): constant-coefficient and homogeneous

$$y' + by = 0$$

  - *homogeneous* means the right-hand side is zero
  - *constant-coefficient* means b is constant
  - the solution is ("by inspection")

$$y(x) = Ae^{-bx}$$

first-order linear review cont.

- special case 2: homogeneous (but otherwise general)

$$y' + P(x)y = 0$$

  - now we need an integrating factor $\mu(x) = e^{Q(x)}$ where $Q(x) = \int P(x)\,dx$ is any antiderivative of $P(x)$
  - multiplying by μ the equation becomes $(\mu(x)y(x))' = 0$
  - thus

$$e^{Q(x)}y(x) = A$$

  - thus the solution is

$$y(x) = Ae^{-Q(x)}$$

  - homogeneous: a multiple of a solution is still a solution

first-order linear review cont.[2]

- general *nonhomogeneous* case: first-order linear

$$y' + P(x)y = f(x)$$

  - need same integrating factor; multiplying by $\mu = e^{Q(x)}$ yields $(\mu(x)y(x))' = \mu(x)f(x)$
  - integrate:

$$e^{Q(x)}y(x) = A + \int_a^x e^{Q(t)} f(t)\,dt$$

    - where $Q(x) = \int P(x)\,dx$ is *any* antiderivative of $P(x)$
    - written to emphasize right side has a free constant A
  - thus the solution is

$$y(x) = Ae^{-Q(x)} + e^{-Q(x)} \int_a^x e^{Q(t)} f(t)\,dt$$

  - solution is the homogeneous solution plus a particular solution

higher-order linear DEs: overview

main idea of §4.1: for nth-order linear equations

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x),$$

new versions of all previous comments in red still apply!

overview cont.

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y \stackrel{*}{=} g(x)$$

1. if $a_n(x) \neq 0$ then we can divide by it:
$$y^{(n)} + b_{n-1}(x)y^{(n-1)} + \cdots + b_1(x)y' + b_0(x)y = f(x)$$
2. easiest case (§4.3) is homogeneous and constant coefficient
$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$
3. for the associated homogeneous equation to $*$,
$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$$
any multiple of, or sum of, solutions is again a solution
4. solutions of $*$ are always solutions of the homogeneous equation plus a particular solution

solutions exist

Theorem 4.1.1

- Consider the linear DE

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$$

  If the functions $a_j(x)$ and $g(x)$ are continuous on some interval, and if $a_n(x) \neq 0$ on that interval, then solutions exist.
- Furthermore, if x_0 is in that interval then there is exactly one solution which satisfies the initial values

$$\begin{aligned} y(x_0) &= y_0 \\ y'(x_0) &= y_1 \\ &\vdots \\ y^{(n-1)}(x_0) &= y_{n-1} \end{aligned}$$

linear, homogeneous, constant-coefficient

- furthermore, linear DEs which are homogeneous and constant-coefficient always have exponential solutions
  - you can always find *at least one solution* $y = e^{mx}$
  - *and* multiples and sums of solutions are solutions
- *example 1*: solve, by trying $y(x) = e^{mx}$, the equation

$$y'' + 4y' - 5y = 0$$

fundamental set of solutions:

general solution:

example 2

- *example 2*: solve, by trying $y(x) = e^{mx}$, the equation

$$y''' + 3y'' - y' - 3y = 0$$

fundamental set of solutions:

general solution:

linear combination

- examples 1 and 2 are from §4.3 (next) but they let me illustrate the language introduced in §4.1 $\longleftarrow$ *read* this!
- for example,

Theorem 4.1.2

If $y_1(x), y_2(x), \dots, y_n(x)$ solve a linear and homogeneous DE

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$$

then any linear combination

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

is also a solution.

- idea: for linear and homogeneous DEs you can form a more general solution from any set of solutions
  - see examples 1 and 2

linear dependence and independence

- a set of functions $\{f_1(x), \ldots, f_n(x)\}$ is *linearly dependent* if you can combine with constants $c_1, \ldots, c_n$, some of which are not zero, and get the zero function:

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

- a set is *linearly independent* if it is not linearly dependent
- *example*:

$$f_1(x) = x^2 + x, \quad f_2(x) = x^2 - x, \quad f_3(x) = 5x$$

are linearly dependent because

$$1 \cdot f_1(x) - 1 \cdot f_2(x) - \frac{2}{5} \cdot f_3(x) = 0$$

example 3

- recall from example 1 that $f_1(x) = e^x$ and $f_2(x) = e^{-5x}$ are solutions to $y'' + 4y' - 5y = 0$
- *example 3*: Find a solution of the initial value problem

$$y'' + 4y' - 5y = 0, \quad y(0) = 2, \quad y'(0) = -3$$

- this calculation works because $\{f_1(x), f_2(x)\} = \{e^x, e^{-5x}\}$ is a linearly-independent set

checking linear independence

- generally it would require linear algebra thinking to check whether a set of functions is linearly independent
- *but* there is a determinant to save you from thinking!
- definition. given functions $f_1(x), \ldots, f_n(x)$ the Wronskian is the determinant where the rows are derivatives:

$$W(f_1, \ldots, f_n) = \det\left(\begin{bmatrix} f_1 & f_2 & \ldots & f_n \\ f_1' & f_2' & \ldots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \ldots & f_n^{(n-1)} \end{bmatrix}\right)$$

- *example 4*: find the Wronskian of $\{e^{-3x}, e^{-x}, e^x\}$

role of the Wronskian

Theorem 4.1.3

Suppose $\{y_1(x), y_2(x), \ldots, y_n(x)\}$ are solutions of a homogeneous linear nth-order differential equation on some interval. Then

- The set of solutions is linearly-independent if and only if the Wronskian $W(y_1, \ldots, y_n)$ is nonzero on the interval.
- If the Wronskian $W(y_1, \ldots, y_n)$ is nonzero at some point on the interval then it is nonzero on the whole interval.

fundamental set

definition.
a set of n linearly-independent solutions $\{y_1(x), y_2(x), \ldots, y_n(x)\}$ of the homogeneous linear nth-order differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = 0$$

is a *fundamental set of solutions*

- once you have a fundamental set then the *general solution* of the above DE is

$$y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

- if you have fewer than n solutions, or they are not linearly independent, then the linear combination *is* a solution, but not fully general

exercise 25 in §4.1

- *exercise #25*: Verify that the functions form a fundamental set of solutions on the interval. Form the general solution.

$$y'' - 2y' + 5y = 0, \qquad \{e^x \cos 2x, e^x \sin 2x\}, \qquad (-\infty, \infty)$$

exercise 27 in §4.1

- *exercise #27*: Verify that the functions form a fundamental set of solutions on the interval. Form the general solution.

$$x^2y'' - 6xy' + 12y = 0, \qquad \{x^3, x^4\}, \qquad (0, \infty)$$

expectations

to learn this material, just listening to a lecture is *not* enough

- *read* section 4.1 in the textbook
- know the meaning/definitions of:

homogeneous	*linearly independent*
nonhomogeneous	*Wronskian*
associated homogeneous equation	*fundamental set of solutions*
linear combination	*general solution*
superposition	*particular solution*
linearly dependent	*complementary function*

- the homogeneous case will be central for a while (§4.3, 4.2)
- more on nonhomogeneous equations in §4.4
- too much new language in §4.1!