3.2 Nonlinear Models

a lecture for MATH F302 Differential Equations

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for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

outline

- like 3.1, section 3.2 has modeling problems
  - *challenging* modeling problems
  - uses separable equations from §2.2: $\frac{dy}{dx} = g(x)h(y)$
- plan for these slides:
  - start with five standard indefinite integrals
    - you did them in calculus I and II
    - . . . but a reminder is appropriate
  - two explanations of where the "logistic equation" comes from
  - solve the logistic equation
  - two exercises from §3.2

integrals you will need

- integral 1 was already used in §3.1:

$$\int \frac{dy}{ay + b} =$$

- integral 2 is sometimes useful:

$$\int \frac{dx}{x^2 + a^2} =$$

integrals, cont.

- integral 3 is the main job in §3.2:

$$\int \frac{dz}{z(a - bz)} =$$

integrals, cont.^{2}

- integral 4 was needed for first-order linear equations (§2.3) and will keep re-appearing in chapter 4 and onward:

$$\int x^n e^{ax}\,dx =$$

$$\int x e^x\,dx =$$

$$\int_0^\infty x^2 e^{-x}\,dx =$$

integrals, cont.[3]

- integral 5 will keep re-appearing in chapter 4 and onward:

$$\int e^{at} \cos bt \, dt =$$

how to remember integrals

- even if you have a good memory, I think it is silly to try to rawly memorize the *results* above
- but *do*
  - try to remember what choices which were made, and *why*
  - remember how to start partial fractions
  - think of integration-by-parts as undoing the product rule
  - $\int x^n e^{ax}\,dx$: integration-by-parts gives a reduction formula
  - $\int e^{at}\cos bt\,dt$: there is a double integration-by-parts trick

logistic equation: explanation 1

Suppose we have a dish with no bacteria, and we plan to supply enough food and water every hour to sustain the needs of N bacteria. Question: What will happen when we introduce a few bacteria ($P_0 \ll N$)?

- *model we know*: $P(t)$ is population, $P(0) = P_0$, $k > 0$,

$$\frac{dP}{dt} = kP$$

  - has solution $P(t) = P_0 e^{kt}$
  - predicts unlimited exponential growth
  - at some point population growth in dish will be limited by food and water

- *better model*: make coefficient get smaller as P gets larger

$$\frac{dP}{dt} = \begin{pmatrix} \text{coefficient which gets} \\ \text{smaller when } P \text{ approaches } N \end{pmatrix} P$$

explanation 1, cont.

- a formula with the desired property:

$$\boxed{\frac{dP}{dt} = k\left(1 - \frac{P}{N}\right)P}$$

- equivalently with $b = k/N > 0$:

$$\boxed{\frac{dP}{dt} = bP\,(N - P)}$$

- previous model was linear; this one is nonlinear
    - but separable!

explanation 1, cont.^{2}

- a formula with the desired property: $\frac{dP}{dt} = bP\,(N - P),\ b > 0$
- draw phase portrait and typical solutions

logistic equation: explanation 2

Suppose a population changes by two mechanisms, namely births (rate is proportional to current population) and deaths by conflicts between the members of the population (rate is proportional to number of interactions, modeled as the square of the population).

- write down the DE, with constants $k > 0$ and $\ell > 0$:

$$\boxed{\frac{dP}{dt} = kP - \ell P^2}$$

- equivalently, using $b = \ell$ and $N = k/\ell$, again write as $\frac{dP}{dt} = bP\,(N - P)$
- look at the last three boxed equations
  - they are all called the *logistic equation*
  - the three forms are equivalent by renaming the constants

logistic equation: solve

- assuming a, b are positive constants, find the general solution:

$$\frac{dP}{dt} = P\left(a - bP\right)$$

logistic equation: verify solution

- show that

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}}$$

solves $\frac{dP}{dt} = P\,(a - bP)$ and $P(0) = P_0$

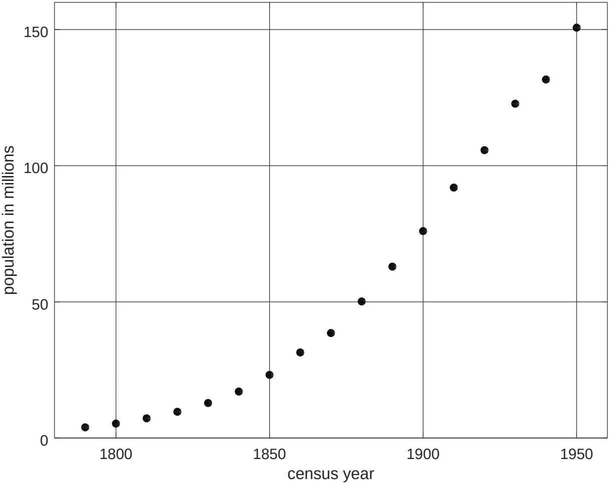
exercise 4 in §3.2

Census data for the United States between 1790 and 1950 are given in the Table on page 102. (a) Construct a logistic population model using the data from 1790, 1850, and 1910. (b) Show a plot comparing actual census population with the population predicted by the model.

- first I put the Table in a Matlab code and plotted it

```
year = 1790:10:1950;    % list of 17 values
pop = [  3.929,   5.308,   7.240,   9.638,  12.866, ...
        17.069,  23.192,  31.433,  38.558,  50.156, ...
        62.948,  75.996,  91.972, 105.711, 122.775, ...
       131.669, 150.697];
plot(year, pop, '.k')
xlabel('census year')
ylabel('population in millions')
axis([1780 1960 0 160]),  grid on
```

exercise 4, cont.



exercise 4, cont.[2]

- we are supposed to construct a logistic model using the 1790, 1850, 1910 populations
- we use those years to determine P_0, a, b in

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}} = \frac{a}{b + (a/P_0 - b)e^{-at}}$$

- let t be number of years after 1790: $P_0 = 3.929$ (millions)
- now 2 equations in 2 unknowns a, b from $P(60) = 23.192$ and $P(120) = 91.972$:

$$23.192 = \frac{a}{b + (a/3.929 - b)e^{-60a}}$$
$$91.972 = \frac{a}{b + (a/3.929 - b)e^{-120a}}$$

- this algebra job is hard!

Problem. Solve 2 equations in 2 unknowns a, b:

$$23.192 = \frac{a}{b + (a/3.929 - b)e^{-60a}}$$
$$91.972 = \frac{a}{b + (a/3.929 - b)e^{-120a}}$$

My solution. I typed into `wolframalpha.com`:

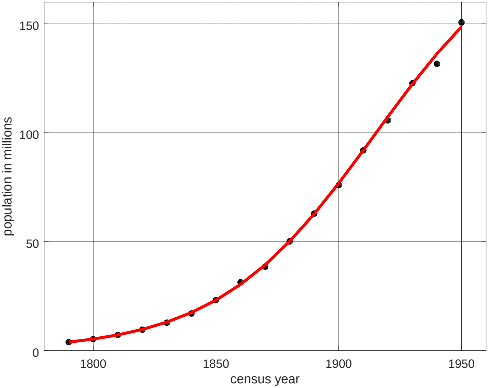
```
solve: 23.192 = a / (b + (a/3.929 - b) e^(-60a)),
       91.972 = a / (b + (a/3.929 - b) e^(-120a))
```

It gave me one *real* answer:

$$a = 0.0313395, b = 0.000158863$$

- checking: with our values of P_0, a, b it returns $P(60) = 23.192$ and $P(120) = 91.972$.
- on the next slide I plot $P(t)$ in red on top of the data; the fit is surprisingly good!

exercise 4, finished



- in our model:

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{a}{b + (a/P_0 - b)e^{-at}} = \frac{a}{b} = 197 \text{ million}$$

- in fact the current US population is definitely larger than that, about 328 million at the start of 2019

exercise 15 in §3.2

Air Resistance *A DE for the velocity v of a falling mass m subjected to air resistance proportional to the square of the velocity is*

$$m\frac{dv}{dt} = mg - kv^2$$

where $k > 0$ and the positive direction is downward.

(a) Solve the equation subject to $v(0) = v_0$.

(b) Determine the terminal velocity. Compare to a phase portrait of the DE.

- note $m > 0$ and $g > 0$
- my first step is to simplify by dividing by m and factoring g:

$$\frac{dv}{dt} = g - \frac{k}{m}v^2 = g\left(1 - \frac{k}{mg}v^2\right)$$

- let $R = \sqrt{k/(mg)}$ to get form: $\frac{dv}{dt} = g\left(1 - R^2v^2\right)$

exercise 15, cont.

- from last slide we want to solve:

$$\frac{dv}{dt} = g\left(1 - R^2 v^2\right) \qquad \text{where } R = \sqrt{\frac{k}{mg}}$$

exercise 15, finished

(b) Determine the terminal velocity. Compare to a phase portrait of the DE.

expectations

- to learn this material, just listening to a lecture is *not* enough!
- *read* section 3.2 in the textbook
  - *read* example 2 on pages 100–101 regarding a chemical equation
  - look for "logistic differential equation" on YouTube or Google
  - do you actually know how to do the integrals at the start of these slides?
  - do Homework 3.2
- up next are second-order linear equations in section 4.1
  - for now we skip section 3.3
  - we will return to it near the end of the course, linking it with material in chapter 8