2.3 Linear Equations

a lecture for MATH F302 Differential Equations

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for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

linear first-order differential equations

a *linear* ordinary differential equation has only a first power on both dy/dx and y, *and* it can be put in the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

or

$$\frac{dy}{dx} + P(x)y = g(x)$$

we can write solutions to such equations in terms of integrals!

examples

linear equation standard form: $\frac{dy}{dx} + P(x)y = g(x)$

examples:

- $$\frac{dy}{dx} + y = x + 3$$

  Here $P(x) = 1$ and $g(x) = x + 3$.

- $$tz' = z + \cos t$$

  which is the same as

  $$\frac{dz}{dt} + \frac{-1}{t} z = \frac{\cos t}{t}$$

  with $P(t) = -1/t$ and $g(t) = \cos t / t$.

not an example

linear equation standard form: $$\frac{dy}{dx} + P(x)y = g(x)$$

not an example:

- $$y\frac{dy}{dx} = x + e^x$$

  this cannot be put in the standard form . . . but it is *separable* (section 2.2)

example 1

before giving general formulas, here's how the method works on an example

- **Example**.

$$\frac{dy}{dx} + y = x + 3$$

solution principle

to solve a first-order, linear ordinary differential equation
$y' + P(x)y = g(x)$ we
multiply by a factor which allows us to *undo* the product rule

recipe

for DE: $y' + P(x)y = g(x)$

1. find $\mu(x)$ so that $\mu'(x) = P(x)\mu(x)$
2. multiply both sides of DE by μ:

$$\mu y' + \mu P y = \mu g$$

3. recognize product rule:

$$\begin{aligned} \mu y' + \mu' y &= \mu g \\ (\mu y)' &= \mu g \end{aligned}$$

4. integrate:

$$\mu(x)y(x) = \int \mu(x)g(x)\,dx$$

5. solve for y:

$$y(x) = \mu(x)^{-1} \int \mu(x)g(x)\,dx$$

integrating factor

formula: the integrating factor $\mu(x)$, which solves

$$\mu'(x) = P(x)\mu(x),$$

has formula

$$\mu(x) = e^{\int P(x)\,dx}$$

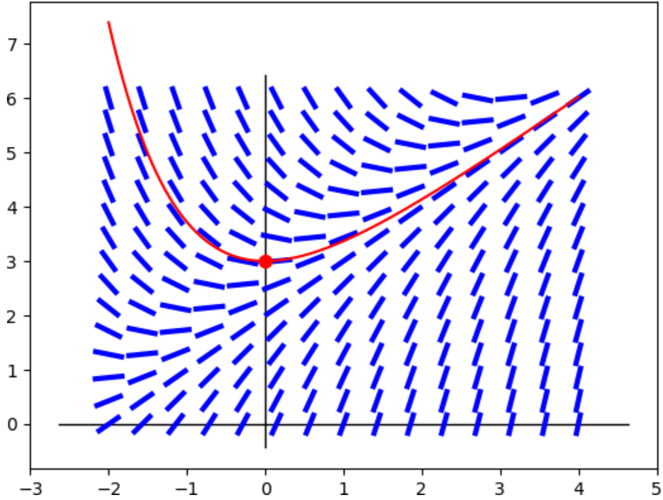
example 2

example: *(has an initial condition)*

$$\frac{dy}{dx} + y = x + 3, \qquad y(0) = 3$$

example:

$$\frac{dy}{dx} + y = x + 3, \qquad y(0) = 3$$

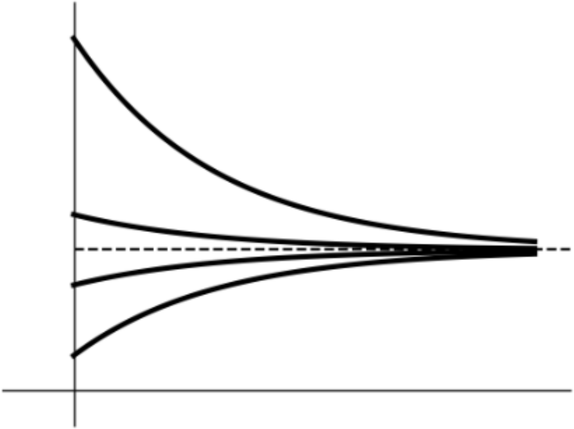


example 3: Newton's law of cooling

example:

$$\frac{dT}{dt} = k(T_m - T), \qquad T(0) = T_0$$

where k, T_m, T_0 are constants



example 3: Newton's law of cooling

example:

$$\frac{dT}{dt} = k(T_m - T), \qquad T(0) = T_0$$

example 4: decaying response to pulses

example: suppose we have a system with solution $s(t)$ which responds to an input $f(t)$:

$$\dot{s} = -ks + f(t), \qquad s(0) = 0$$

suppose the input $f(t)$ is a sequence of pulses as in this graph:

example 4: decaying response to pulses

example:

$$\dot{s} = -ks + f(t), \qquad s(0) = 0$$

example 4: decaying response to pulses

example:

$$\dot{s} = -ks + f(t), \qquad s(0) = 0$$

solution:

$$\begin{aligned} s(t) &= e^{-kt} \int_0^t e^{k\tau} f(\tau)\, d\tau \\ &= \int_0^t e^{-k(t-\tau)} f(\tau)\, d\tau \end{aligned}$$

interpret this?

example 5: careful (cute) algebra

example:

$$x^2 y' + x(x+2)y = e^x$$

standard expectations

to learn this material, just listening to a lecture is *not* enough

- please *read* section 2.3 in the textbook
- please *do* the Homework for section 2.3
- search "linear ODEs" at YouTube to see more examples