# 2.1 Solution Curves
# (direction fields and phase portraits)

a lecture for MATH F302 Differential Equations

Ed Bueler, Dept. of Mathematics and Statistics, UAF

Fall 2023

for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

# meaning of a differential equation

- let us start over on the meaning of a differential equation (DE)
- for the DE
$$\frac{dy}{dx} = f(x, y)$$
  1. the left side is the *slope* of the solution $y(x)$
  2. given a point $(x, y)$, the right side computes a number $f(x, y)$
- thus a first-order DE says:

$$\begin{array}{c} \text{the slope of the} \\ \text{solution } y(x) \end{array} \overset{\text{equals}}{=} \begin{array}{c} \text{a known function of} \\ \text{the location } (x, y) \end{array}$$

# picturing a DE

- the first-order DE $\dfrac{dy}{dx} = f(x, y)$ says:

$$\boxed{\begin{array}{ccc} \text{the slope of the} & & \text{a known function of} \\ & \text{equals} & \\ \text{solution } y(x) & & \text{the location } (x, y) \end{array}}$$

- this means that

  we can *draw* a picture of the DE itself

  by drawing the slope $m = \frac{dy}{dx}$ at each point $(x, y)$
- we can do this whether or not we can do the calculus/algebra to find a solution formula $y(x)$

## direction field

- from the first-order DE $\dfrac{dy}{dx} = f(x, y)$ we can create a *direction field* or *slope field*:
  1. generate a grid of point in the $x$,$y$ plane
  2. for each point, draw a short line segment with the slope given by $f(x, y)$ at that point

- *Example.* By hand, draw a direction field:

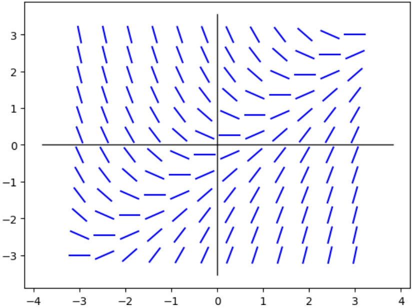
$$\frac{dy}{dx} = x - y$$

  on the square
  $-3 \leq x \leq 3$,
  $-3 \leq y \leq 3$

## computers are useful

- I *happily* acknowledge that this is a job for a computer!
  - see the Codes tab at the public site
  - I will use a Python function `dirfield()`
- *Example.* Use a computer to draw a direction field for $\frac{dy}{dx} = x - y$ on the square $-3 \leq x \leq 3, -3 \leq y \leq 3$

*Solution*:




```
def f(x,y):  return x - y
dirfield(f,[-3,3,-3,3],mx=12,my=12)
```

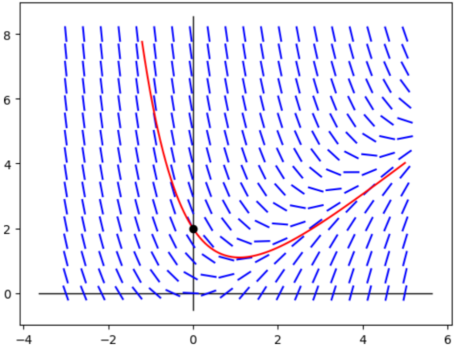
⟵ *Python code*

- recall that we are often solving initial value problems (IVPs)
- *next main idea*: one can *see* the solution to an ODE IVP by plotting the initial point in the plane and then following the direction field from that point

- *Example.* Use the direction field for $\frac{dy}{dx} = x - y$ to sketch the solution $y(x)$ of

$$\frac{dy}{dx} = x - y,\ y(0) = 2$$

  - soon: methods in §2.3 will give a formula for $y(x)$

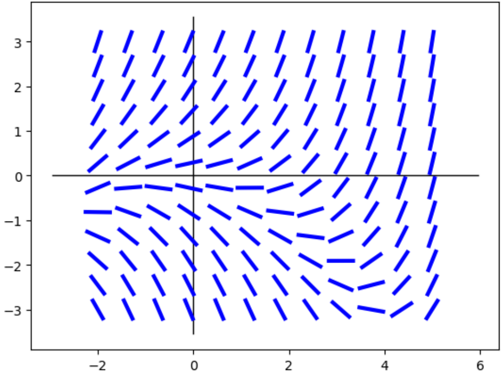


# exercise 9 in §2.1

**9**. *Use computer software to obtain a direction field for the given differential equation. By hand, sketch an approximate solution curve passing through each of the given points.*

$$\frac{dy}{dx} = 0.2x^2 + y$$

**(a)** $y(0) = \frac{1}{2}$
**(b)** $y(2) = -1$




```
def f(x,y):  return 0.2*x**2 + y
dirfield(f,[-2,5,-3,3],mx=12,my=12)
```

# two topics in §2.1

- there are two equally-important topics in §2.1:
  - direction fields for 1st-order DEs
  - autonomous 1st-order DEs, and their phase portraits
- both topics are about *picturing DEs*

- next: "autonomous" describes a special case where we can draw a simpler picture

## autonomous first-order DEs

- *definition*. a first-order differential equation is *autonomous* if the function does not depend on the independent variable:

$$\frac{dy}{dx} = f(y)$$

  - "autonomous" means "independent of control"
  - ...above DE is not directly controlled by input variable $x$
  - note that the *solution* $y(x)$ is still a function of $x$

- a big idea: fundamental laws of nature, in which the independent variable is the time $t$, are autonomous DEs

## autonomous first-order DEs

- *definition*. a first-order differential equation is *autonomous* if the function does not depend on the independent variable:

$$\frac{dy}{dx} = f(y)$$

- Example.

$$\frac{dy}{dx} = \sqrt{\sin(y)} \quad \text{is autonomous}$$

- Example.

$$\frac{dy}{dx} = x - y \quad \text{is } \textit{not} \text{ autonomous}$$

## classification of first-order DEs

- we will see that "autonomous" also means "easier to visualize," but *not* necessarily easier to solve
- using definitions from sections 1.1 and 2.1 we already have a *classification* of first-order DEs:

| | autonomous | nonautonomous |
|---|---|---|
| linear | $y' + c\,y = d$ | $y' + P(x)y = g(x)$ |
| nonlinear | $y' = f(y)$ | $y' = f(x, y)$ |

  - all of these can be written as $y' = f(x, y)$, right?

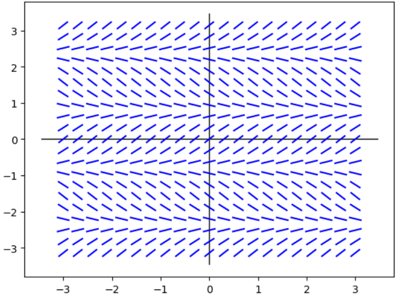
# picturing autonomous DEs

- the direction field of an autonomous DE has redundancies
  - since $f(y)$ only depends on $y$, the slope is the same for all $x$ values

- Example. Consider

$$\frac{dy}{dx} = \cos(2y)$$

Use a computer to draw the direction field for $x \in [-3, 3]$ and $y \in [-\pi, \pi]$. Then draw the phase portrait.




- the one-dimensional *phase portrait* is a simplified form of the direction field
  - a.k.a. *phase line*

## critical points of autonomous DEs

- consider an autonomous first-order DE: $\frac{dy}{dx} = f(y)$
- a value $y = c$ is called a *critical point* if $f(c) = 0$
  - a.k.a. *equilibrium point* or *stationary point*
- if $y = c$ is a critical point then $y(x) = c$ is a solution!
- Example. $y = \frac{\pi}{4}$ is a critical point *and a solution* of

$$\frac{dy}{dx} = \cos(2y)$$

# phase portrait example

- Example. By hand, sketch the phase portrait of

$$\frac{dz}{dt} = z^2 + z^3$$

and show all critical points. Then sketch the graph of solutions to the ODE IVP with the following initial values.

**(a)** $z(0) = 1$

**(b)** $z(0) = -1/2$

**(c)** $z(0) = -1$

**(c)** $z(0) = -2$

## to draw a phase portrait of an autonomous equation

for:

$$\frac{dy}{dx} = f(y)$$

summary:

- draw the $y$-axis vertically
- solve $f(y) = 0$ for the critical points, and mark them
- between critical points, evaluate the sign of $f(y)$
- draw an up or down arrow accordingly

*easy*!

# classifying critical points

- a critical point $y = c$ of an autonomous differential equation $\dfrac{dy}{dx} = f(y)$ is
  - *attracting* or *asymptotically stable* if

    $$\lim_{x \to \infty} y(x) = c \tag{$*$}$$

    for all initial points $(x_0, y_0)$ where $y_0$ is close to $c$,
  - *semi-stable* if $(*)$ only happens for $y_0$ on one side of $c$, and
  - *repelling* or *asymptotically unstable* otherwise

## example 2

- Example. Find and classify the critical points of

$$\frac{dy}{dx} = \cos(2y)$$

## example 3

- Example. Find and classify the critical points of

$$\frac{dz}{dt} = z^2 + z^3$$

# exercise 27 in §2.1

**27**. *Find the critical points and phase portrait. Classify each critical point as asymptotically stable, unstable, or semi-stable. By hand, sketch typical solution curves in the regions in the $xy-$plane determined by the graphs of the equilibrium solutions.*

$$\frac{dy}{dx} = y\ln(y+2)$$

# exercise 40 in §2.1

**40**. *The autonomous differential equation*

$$m\frac{dv}{dt} = mg - kv,$$

*where $k$ is a positive constant and $g$ is the acceleration due to gravity, is a model for the velocity $v$ of a body of mass $m$ that is falling under gravity. The term $-kv$, which is air resistance, implies that the velocity will not increase without bound as $t$ increases. Use a phase portrait to find the limiting, or terminal velocity of the body.*

## looking ahead: next two sections 2.2, 2.3

- the first four sections of the textbook (1.1, 1.2, 1.3, 2.1) are about the *meaning* of differential equations
  - such meaning is the important take-home message from a course in differential equations!
- but for the next few sections we will address how to find formulas for solutions $y(x)$
- looking ahead to the next two sections:

| | autonomous | nonautonomous | |
|---|---|---|---|
| linear | $y' + c\,y = d$ 2.2,2.3 | $y' + P(x)y = g(x)$ 2.3 | |
| nonlinear | $y' = f(y)$ 2.2 | separable | nonseparable |
| | | $y' = g(x)h(y)$ 2.2 | $y' = f(x, y)$ |

## programming . . . this is not a CS class

- you *don't* have to know programming to do this class
- . . . but interacting with a computer is obligatory for some homework!
- for such homework problems you must either embrace some coding or seek-out tools such as desmos or Wolfram alpha which allow you to do particular computer jobs like generating direction fields
- I will generally show a few lines of Matlab or Python when there is a computer-suitable job
- . . . *and* I'll link to some programming-free tools

**expectations**: to learn this material, just listening to a lecture is *not* enough

- please *read* section 2.1 in the textbook
  - note there is a large new vocabulary in this section, namely the language of *qualitative* differential equations
  - I did not cover "translation property" on page 43; read that!
- please *do* the Homework for section 2.1
- the visualization jobs in section 2.1 are a topic for which youtube videos etc. can be excellent resources