1.3 Differential Equations as Mathematical Models

a lecture for MATH F302 Differential Equations

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for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

DEs as models

- I have already made a big deal of differential equations as models in previous slides . . .
- the goal of each example and exercise in §1.3 is to write down a differential equation as a model of some situation
  - often don't need to solve the DE
  - generally first-order DE
- for section §1.3 my plan is:
  - I will work-through 4 exercises from the section here
  - *you* will actually read the examples in the section
  - on Homework 1.3 *you* will do some exercises from the section

exercise 2 in §1.3

2. *The population model given in (1) fails to take death into consideration: the growth rate equals the birth rate. In another model of a changing population of a community it is assumed that the rate at which the population changes is a net rate—that is, the difference between the rate of births and the rate of deaths in the community. Determine a model for the population* $P(t)$ *if both the birth rate and the death rate are proportional to the population present at time* $t > 0$.

- the population model in (1) is simply that the rate of change of population is proportional to the population:

$$\frac{dP}{dt} = kP$$

exercise 2 cont.

2. *The population model given in (1) fails to take death into consideration: the growth rate equals the birth rate. In another model of a changing population of a community it is assumed that the rate at which the population changes is a net rate—that is, the difference between the rate of births and the rate of deaths in the community. Determine a model for the population $P(t)$ if both the birth rate and the death rate are proportional to the population present at time $t > 0$.*

- this exercise asks for "another model" where "both the birth rate and death rate are proportional" to $P(t)$
  - $P(t) =$ "the population present at time $t > 0$"
- in the new model we want $\frac{dP}{dt}$ to be the *net* rate
- the net rate is "the difference between the rate of births and the rate of deaths"

exercise 2 cont.

- the rate at which the population changes is net rate:

$$\frac{dP}{dt} = (\text{rate of births}) - (\text{rate of deaths})$$

- both the birth rate and death rate are proportional to $P(t)$:

$$(\text{rate of births}) = k_b P$$
$$(\text{rate of deaths}) = k_d P$$

where k_b, k_d are two new *positive* constants

exercise 2 cont. cont.

- the new model combines the stuff on last slide:

$$\frac{dP}{dt} = k_b P - k_d P$$

- show this new model is really the old model (1):

- *conclusion*. we see that (1) *already allows* births and deaths, with $k = k_b - k_d$

- please go back and actually *read* the "Population Dynamics" example on page 23

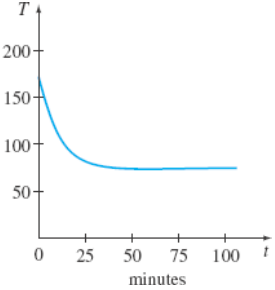
exercise 5 in §1.3

5. *A cup of coffee cools according to Newton's law of cooling. Use data from the graph of temperature $T(t)$ [below] to estimate the constants T_m, T_0, and k in a model of the form of a first order initial-value problem: $dT/dt = k(T - T_m)$, $T(0) = T_0$.*

- Newton's law of cooling says that an object with temperature $T(t)$ warms or cools at a rate proportional to the difference between $T(t)$ and the ambient temperature T_m:

$$\frac{dT}{dt} = k(T - T_m)$$

- solve by extracting numbers from the graph:



exercise 21 in §1.3

21. *A small single-stage rocket is launched vertically as shown. Once launched, the rocket consumes its fuel, and so its total mass $m(t)$ varies with time $t > 0$. If it is assumed that the positive direction is upward, air resistance is proportional to the instantaneous velocity v of the rocket, and R is the upward thrust or force, then construct a mathematical model for the velocity $v(t)$ of the rocket.*

- hint 1: when the mass is changing with time, Newton's law is

  $$F = \frac{d}{dt}(mv) \qquad (17)$$

  where F is the net force on the body and mv is the momentum

- hint 2: on page 27 there is a model for air resistance used in equation (14): $F_2 = -kv$

exercise 21, cont.

- collect the forces to get the net force:

$$F =$$

- now we can write down the model:

exercise 10 in §1.3

10. *Suppose that a large mixing tank initially holds 300 gallons of water in which 50 pounds of salt have been dissolved. Another brine solution is pumped into the tank at a rate of 3 gallons per minute [gal/min], and when the solution is well-stirred it is then pumped out at a slower rate of 2 gal/min. If the concentration of the solution entering is 2 pounds per gallon [lb/gal], determine a differential equation for the amount of salt* $A(t)$ *in the tank at time* $t > 0$.

- $A(t)$ is amount of salt in pounds [lb]; what is $A(0)$?
- what is $V(t)$, the total solution volume?
- write down the differential equation for $\frac{dA}{dt}$:

exercise 10, extended and fully-solved

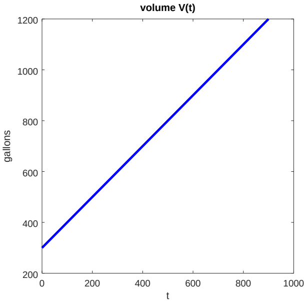
- what is a function $A(t)$ satisfying the ODE IVP?:

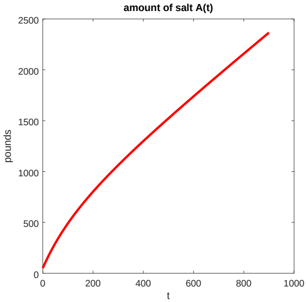
$$\frac{dA}{dt} = 6 - \frac{2}{300+t}A, \quad A(0) = 50$$

- one may verify that

$$A(t) = 2(300+t) - 550\left(\frac{300}{300+t}\right)^2$$

  - get it using methods in §2.3





standard expectations

expectations: to learn this material, just listening to a lecture is *not* enough

- please *read* section 1.3 in the textbook
  - for instance, actually *read* the "Mixtures" example on p. 25 and the "Falling Bodies and Air Resistance" example on p. 27
- please *do* the Homework for section 1.3