1.2 Initial-Value Problems

a lecture for MATH F302 Differential Equations

Ed Bueler, Dept. of Mathematics and Statistics, UAF

Fall 2023

for textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

main purpose of DEs

- the main purpose of differential equations (DEs) in science and engineering:

DEs can make predictions

- two things are needed to make a prediction:

precise description of rate of change	=	differential equation
knowledge of current state	=	initial conditions

- sections 1.1 and 1.2 introduce these two things

prediction models

- professionals are skeptics about using math for predictions, and DEs do *not* “know the future”, but they are *models* which are *capable* of prediction
- next two slides are examples

don’t worry about understanding the specific equations on the next two slides

an amazingly-accurate prediction model

- Newton's theory of gravitation gives remarkably-accurate predictions of planets, satellites, and space probes
- the DEs at right are Newton's model of many particles interacting by gravity
- ... a system of coupled, nonlinear, 2nd-order ODEs for the position $\mathbf{r}_i$ of each object with mass m_i

$$\frac{d^2\mathbf{r}_i}{dt^2} = G \sum_{j \neq i} \frac{m_i m_j}{|\mathbf{r}_j - \mathbf{r}_i|^3}(\mathbf{r}_j - \mathbf{r}_i)$$

en.wikipedia.org/wiki/Equations_of_motion

a pretty-good prediction model

- *weather prediction* uses Euler's fluid model of the atmosphere
- ...a system of PDEs; equations at right
- predictions have been refined by comparing prediction to what actually happened
- modern: about 6 days of good/helpful predictions

Euler equation(s) (*conservation form, for thermodynamic fluids*)

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \mathbf{j}\\ S\end{pmatrix} + \nabla\cdot\begin{pmatrix}\mathbf{j}\\ \frac{1}{\rho}\mathbf{j}\otimes\mathbf{j}+p\mathbf{I}\\ S\frac{\mathbf{j}}{\rho}\end{pmatrix} = \begin{pmatrix}0\\ \mathbf{f}\\ 0\end{pmatrix}$$

`en.wikipedia.org/wiki/Euler_equations_(fluid_dynamics)`

don't worry: this course is about ODEs and not systems of PDEs

what kind of student are you?

- did you ignore the last few slides because all you want to know is how to do the homework problems quicker?
- I observe that
  - better students choose to be curious and interested
  - better students have at least some tentative trust that teachers are providing an ultimately easier and clearer path

example 1

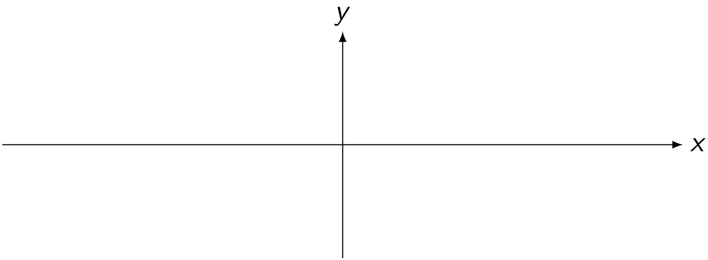
- *example*: here is the single most important ODE:

$$y' = y$$

  - it is first-order and linear
- just by thinking one can write down all of its solutions:

$y(x) =$

- graph and label several particular solutions:

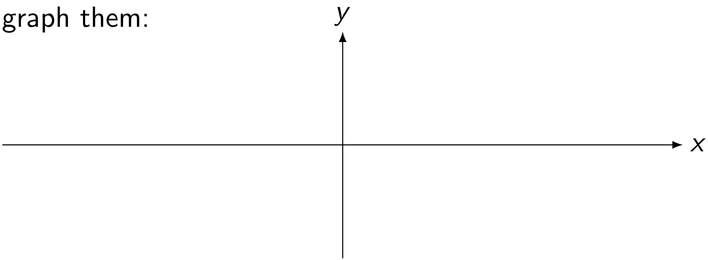


example 1, cont.

- initial conditions "pick out" one prediction (solution) from all the solutions of a differential equation
- *for example*, fill in the table:

ODE IVP	solution
$y' = y,\ y(0) = 3$	$y(x) =$
$y' = y,\ y(3) = -1$	$y(x) =$
$y' = y,\ y(-1) = 1$	$y(x) =$

- graph them:



example 2

- as we will show later,

$$y(x) = c_1 \sin(3x) + c_2 \cos(3x)$$

describes all the solutions of

$$y'' + 9y = 0$$

- *example*. solve this 2nd-order linear ODE IVP:

$$y'' + 9y = 0, \quad y(0) = 2,\ y'(0) = -1$$

example 3

- *example*. now solve this 2nd-order linear ODE IVP:

$$y'' + 9y = 0, \quad y(2) = -3, \; y'(2) = 0$$

example 4

- *example*. now solve this problem:

$$y'' + 9y = 0, \quad y(0) = 0,\ y(1) = 3$$

- the above has *boundary* conditions at $x = 0$ and $x = 1$
  - *not* an IVP
  - potentially problematic; for example,
    $y'' + 9y = 0$, $y(0) = 0$, $y(\pi/3) = 3$ has *no* solutions

general IVP

- in Math 302 we will stick to *initial* conditions
  - *not* boundary conditions
- the general form of an initial-value problem for an ordinary differential equation (ODE IVP):

$$\begin{aligned}
\frac{d^n y}{dx^n} &= f(x, y, y', \dots, y^{(n-1)}) \\
y(x_0) &= y_0 \\
y'(x_0) &= y_1 \\
&\vdots \\
y^{(n-1)}(x_0) &= y_{n-1}
\end{aligned}$$

  - this is equation (1) at the start of section 1.2

main idea

- as suggested earlier, the main idea is that an ODE IVP is a *model capable of prediction*
  - law of how things change (= the DE) plus the current state (= the initial values)
- to make a prediction, two questions need "yes" answers:
  1. does a solution of the ODE IVP exist?
  2. is there only one solution of ODE IVP?
- people often say "is the solution unique?" for the second question

theorem about main idea

- for nicely-behaved first-order ODE IVPs, the answer to both questions is “yes”!
  - “nicely-behaved” means that the differential equation is continuous enough
- consider the first-order ODE IVP

$$(*) \qquad y' = f(x, y), \quad y(x_0) = y_0$$

Theorem (1.2.1)

Let R be a rectangle in the xy plane that contains (x_0, y_0) in the interior. Suppose that $f(x, y)$ in $(*)$ is continuous and the $\frac{\partial f}{\partial y}(x, y)$ is also continous. Then there is exactly one solution to ODE IVP, but it may only be defined for a short part of the x-axis around x_0, i.e. on an open interval $(x_0 - h, x_0 + h)$.

an example

- the last slide was "mathy"; an example helps give meaning
- *example*. verify that both $y(x) = 0$ and $y(x) = cx^{3/2}$, for some nonzero c, solve the ODE IVP

$$y' = y^{1/3}, \quad y(0) = 0$$

- in the above example $\frac{\partial f}{\partial y} = \frac{1}{3} y^{-2/3}$
    - it is not continuous on any rectangle around $(0, 0)$
- as in the theorem on the last slide, this example shows you need $f(x, y)$ to be a bit "nice" to make a single prediction

conclusion

- the main idea of section 1.2 is in this slogan:

  if you add initial condition(s) to a differential equation then you can get a single solution

- Theorem 1.2.1 says this is actually true of first-order ODE IVPs ($y' = f(x, y)$) with a single initial value ($y(x_0) = y_0$) as long as the function f is nice
- *important notes*:
  - to use the language of prediction, we would call $x < x_0$ the "past" and $x > x_0$ the "future"
  - for nth-order ODEs (second-order, third-order, etc.) the Theorem does not directly apply, but we expect to need n numbers to give adequate initial conditions/values

standard expectations

expectations: to learn this material, just listening to a lecture is *not* enough

- please *read* section 1.2 in the textbook
  - and browse section 1.3
- please *do* the Homework for section 1.2
- see this Wikipedia page for more on Theorem 1.2.1:
  `en.wikipedia.org/wiki/Picard-Lindelöf_theorem`