1.1 Definitions and Terminology

a lecture for MATH F302 Differential Equations

Ed Bueler, Dept. of Mathematics and Statistics, UAF

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textbook: D. Zill, *A First Course in Differential Equations with Modeling Applications*, 11th ed.

basic

- a *differential equation* is an equation with a derivative somewhere in it

definitions

- idea: a differential equation contains an *unknown function* which we want to find
- an *ordinary differential equation* (ODE) uses ordinary derivatives (as in calculus I and II)
  - primes ($y' = dy/dx$) or dots ($\dot{y} = dy/dt$) are often used to denote ordinary derivatives
  - examples of ODEs:

$$\frac{dy}{dx} = x + y^2 \qquad y(x) \text{ is unknown function}$$

$$y' = x + y^2 \qquad \ldots \text{ exactly the same}$$

$$\frac{d^2u}{dt^2} = -cu \qquad u(t) \text{ is unknown function}$$

$$\ddot{u} = -cu \qquad \ldots \text{ exactly the same}$$

  - the unknown function in an ODE depends on one variable

contrast with PDEs

- MATH 302 is about ODEs
- ...but there are also *partial differential equations* (PDEs)
  - subscripts are often used to denote partial derivatives
  - examples:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad u(t,x) \text{ is unknown function}$$

$$u_{tt} = c^2 u_{xx} \qquad \ldots \text{ exactly the same PDE}$$

$$w_t = k(w_{xx} + w_{yy}) \qquad w(t,x,y) \text{ is unknown function}$$

  - the unknown function in a PDE depends on more than one variable
- do not worry about PDEs!
  - they are covered in MATH 432
  - I am just explaining here why people say "ordinary"

order

- the *order* of a differential equation is the maximum number of derivatives
  - order has nothing to do with powers or exponentials
  - most of the differential equations in MATH 302 have order 1 or order 2
- examples:
  - $y' = x + y^2$ has order one
  - $\ddot{u} = -cu$ has order two
    - c is just a constant in this context
  - $y^3 + \frac{d^4y}{dx^4} = \left(\frac{d^2y}{dx^2} + \sin x\right)^5$ has order four

two main operations on ODEs

- there is more terminology to come . . . but let's *do* something
- two common operations with differential equations are
  - *verify* that a given function is a solution
  - *construct* a solution ("solve the differential equation")
- *example*: verify that $y(x) = \sin(3x)$ solves $y'' + 9y = 0$

- *example*: construct a solution to $y' = y^2$

visualization of solutions

- a given differential equation generally has many solutions
- *example (a)*: show that for any value of the *parameter* A the function $y(x) = Ae^{-x^2/2}$ solves $\frac{dy}{dx} = -xy$
- *example (b)*: sketch several *particular* solutions from *(a)*

linear

- back to terminology
- a differential equation is *linear* if it can be written as a sum with only first powers on the unknown function and its derivatives
- examples:
  - $3y'' - 7y' + 8y = \sin x$ is linear because it is in the form

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

  - $x\frac{y'}{y} = x^2 + 5$ is linear because it it *can* be written in the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

(set: $a_1(x) = x,\ a_0(x) = -x^2 - 5,\ g(x) = 0$)

nonlinear

- linear differential equations are special and easier
  - nature has been generous by allowing good models of surprisingly-many situations to be built using linear differential equations
- most differential equations are *nonlinear* which only means they are not linear
- examples:
  - $y' = y^2$ is nonlinear
  - $y'' + \sin y = 0$ is nonlinear
- one way to understand MATH 302:
  - we will be able to solve *some* nonlinear ODEs
  - we will be *systematic* about solving linear ODEs

implicit solution

- first, remember *implicit differentiation*
  - *example*: find dy/dx if $x \sin y + y^2 = \ln x$

- the statement "verify [this implicitly-defined function] is a solution of [this differential equation]" asks for implicit differentiation
  - *example*: verify $y = e^{xy}$ defines a solution of $(1 - xy)y' = y^2$

extras

the book mentions more terminology; none of this is terribly important, but it is used in the rest of the semester:

page 5 *normal form* means the highest derivative is on the left; the normal form of $y' - y^2 = 0$ is $y' = y^2$, and the normal form of $u'' + 9u = e^t$ is $u'' = e^t - 9u$

page 7 a function $y(x)$ can be discontinuous but when the book uses the term *solution* for $y(x)$ then it solves a differential equation *and* we assume it is continuous on some interval

page 11 a function like $F(x) = \int_a^x g(t)\,dt$ is an *integral-defined function*; the most important thing to know is that the derivative is easy: $F'(x) = g(x)$

- see Wed. 30 August worksheet

standard expectations

expectations: to learn this material, just listening to a lecture is *not* enough

- please *read* section 1.1 in the textbook
  - and browse section 1.2
- please *do* the Homework for section 1.1
- please *look around* the bueler.github.io/math302 website
- please find other videos and related content!