1.2 Initial-Value Problems
a lesson for MATH F302 Differential Equations

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main purpose of DEs

• the main purpose of differential equations (DEs) in science and engineering:

DEs are models which are capable of prediction

• two things are needed to make a prediction:

  precise description
  of rate of change

  ⇐⇒

  differential equation

  knowledge
  of current state

  ⇐⇒

  initial conditions

• sections 1.1 and 1.2 introduce these two things
prediction models

- all professionals are skeptics about using math for predictions
- DEs do not “know the future”
- ...but they are models which are capable of prediction
- next two slides are examples

*don’t worry: about understanding the specific equations on the next two slides*
an amazingly-accurate real prediction model

- Newton’s theory of gravitation gives remarkably-accurate predictions of planets, satellites, and space probes.
- The DEs at right are Newton’s model of many particles interacting by gravity.
- ...a system of coupled, nonlinear, 2nd-order ODEs for the position $r_i$ of each object with mass $m_i$.
- Adding corrections for relativity makes these predictions practically perfect.

\[
\frac{d^2 r_i}{dt^2} = G \sum_{j \neq i} \frac{m_i m_j}{|r_j - r_i|^3} (r_j - r_i)
\]

en.wikipedia.org/wiki/Equations_of_motion
a pretty-good real prediction model

- *weather prediction* uses Euler’s fluid model of the atmosphere
- ... a system of PDEs; equations at right
- predictions have been refined by comparing prediction to what actually happened
- ... now we get about 6 days of good/helpful predictions

\[ \begin{align*}
\frac{\partial}{\partial t} & \begin{pmatrix} \rho \\ \mathbf{j} \\ S \end{pmatrix} + \nabla \cdot \begin{pmatrix} \frac{1}{\rho} \mathbf{j} \otimes \mathbf{j} + p \mathbf{I} \\ S \frac{\mathbf{j}}{\rho} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f} \\ 0 \end{pmatrix}
\end{align*} \]

en.wikipedia.org/wiki/Euler_equations_(fluid_dynamics)

*don’t worry:* this course is about ODEs and not systems of PDEs
what kind of student are you?

- did you skip the last few slides because you want to know how to do the homework problems quicker?
- I observe that
  - better students choose to be curious and interested
  - better students have at least some tentative trust that teachers are seeking an easy path through the whole subject

- in any case, there will be homework about DE models in section 1.3 . . . coming soon
example 1

- *example*: here is the single most important ODE:

\[ y' = y \]

  - it is first-order and linear

- just by thinking you can write down all of its solutions:

\[ y(x) = \]

- please graph and label several particular solutions:
initial conditions “pick out” one prediction (solution) from all the solutions of a differential equation

*for example*, fill in the table:

<table>
<thead>
<tr>
<th>ODE IVP</th>
<th>solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y' = y, \ y(0) = 3$</td>
<td>$y(x) =$</td>
</tr>
<tr>
<td>$y' = y, \ y(3) = -1$</td>
<td>$y(x) =$</td>
</tr>
<tr>
<td>$y' = y, \ y(-1) = 1$</td>
<td>$y(x) =$</td>
</tr>
</tbody>
</table>

graph them:
example 2

- as we will show later,

\[ y(x) = c_1 \sin(3x) + c_2 \cos(3x) \]

is the general solution of (all of the solutions of)

\[ y'' + 9y = 0 \]

- example. solve this 2nd-order linear ODE IVP:

\[ y'' + 9y = 0, \quad y(0) = 2, \ y'(0) = -1 \]
example 3

- example. now solve this 2nd-order linear ODE IVP:

\[ y'' + 9y = 0, \quad y(2) = -3, \quad y'(2) = 0 \]
example 4

- example. now solve this problem:

\[ y'' + 9y = 0, \quad y(0) = 0, \quad y(1) = 3 \]

- the above has *boundary* conditions at \( x = 0 \) and \( x = 1 \)
  - *not* an IVP
  - potentially problematic; for example,
    \[ y'' + 9y = 0, \quad y(0) = 0, \quad y(\pi/3) = 3 \]
    has *no* solutions
in Math 302 we will stick to *initial* conditions
  ○ *not* boundary conditions

the general form of an initial-value problem for an ordinary differential equation (ODE IVP):

\[
\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})
\]
\[
y(x_0) = y_0
\]
\[
y'(x_0) = y_1
\]
\[
\vdots
\]
\[
y^{(n-1)}(x_0) = y_{n-1}
\]

○ this is equation (1) at the start of section 1.2
• as suggested earlier, the main idea is that an ODE IVP is a model capable of prediction
  ○ law of how things change (= the DE) plus the current state (= the initial values)
• to have a prediction, two questions need “yes” answers:
  1. does a solution of the ODE IVP exist?
  2. is there only one solution of ODE IVP?
• people often say “is the solution unique?” for the second question
for nicely-behaved first-order ODE IVPs, the answer to both questions is “yes”!
  - “nicely-behaved” means that the differential equation is continuous enough

- consider the first-order ODE IVP

\[
(*) \quad y' = f(x, y), \quad y(x_0) = y_0
\]

**Theorem (1.2.1)**

Let \( R \) be a rectangle in the \( xy \) plane that contains \((x_0, y_0)\) in the interior. Suppose that \( f(x, y) \) in (*) is continuous and the \( \frac{\partial f}{\partial y} (x, y) \) is also continuous. Then there is exactly one solution to ODE IVP, but it may only be defined for a short part of the \( x \)-axis around \( x_0 \), i.e. on an open interval \((x_0 - h, x_0 + h)\).
an example

- the last slide was “mathy”; an example helps give meaning
- example. verify that both \( y(x) = 0 \) and \( y(x) = cx^{3/2} \), for some nonzero \( c \), solve the ODE IVP

\[
y' = y^{1/3}, \quad y(0) = 0
\]

- in the above example \( \frac{\partial f}{\partial y} = \frac{1}{3}y^{-2/3} \)
  - it is not continuous on any rectangle around \((0, 0)\)
- the theorem on the last slide is true but this example shows you do need \( f(x, y) \) to be nice
• the main idea of section 1.2 is in this slogan:

\textit{if you add initial condition(s) to a differential equation then you can get a single solution, which can be used to predict}

• Theorem 1.2.1 says this is actually true of first-order ODE IVPs ($y' = f(x, y)$) with a single initial value ($y(x_0) = y_0$) as long as the function $f$ is nice

• important notes:
  ○ to use the language of prediction, we would call $x < x_0$ the “past” and $x > x_0$ the “future”
  ○ for $n$th-order ODEs (second-order, third-order, etc.) the Theorem does not directly apply, but we expect to need $n$ numbers to give adequate initial conditions/values
expectations: to learn this material, just watching this video is not enough; also

- *read* section 1.2 in the textbook
- *do* the WebAssign exercises for section 1.2
- *think* about these ideas

- see this page for more on Theorem 1.2.1: en.wikipedia.org/wiki/Picard-Lindelöf_theorem