Name: ______________________________ / 25

35 minutes maximum. No aids (book, calculator, etc.) are permitted. Show all work; use proper notation for full credit. Answers should be in reasonably-simplified form. 25 points possible.

1. [8 points] Solve the following initial value problem:

$$y'' + 5y' + 4y = 4x + 5, \quad y(0) = 0, y'(0) = 0$$

$y(x) =$

2. [9 points] A force of 400 Newtons stretches a spring 2 meters. A mass of 50 kilograms is attached to the end of the spring and then is initially released from from the equilibrium position with an upward velocity of 10 m/s.

a) The above describes an undamped mass-spring system. Find the equation of motion.

$x(t) =$

b) Suppose we add a damping force $-250\,dx/dt$, that is, with $\beta = 250$. What is the new differential equation? Find the general solution; you may ignore the initial conditions for this part.

$x(t) =$

c) Going back to the undamped mass-spring system in part ***a)***, suppose we apply an external force $f(t) = 50e^{-3t}$. Find the general solution; again you may ignore the initial conditions for this part.

$x(t) =$

3. [8 points] Recall the rules for a 2nd-order, linear, constant-coefficient, and homogeneous differential equation

$$ay'' + by' + cy = 0$$

which has auxiliary equation $am^2 + bm + c = 0$ from substituting $y(x) = e^{mx}$.

(i) If the roots m_1, m_2 of the auxiliary equation are real then $y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$.

(ii) If the root $m_1 = m_2$ of the auxiliary equation is repeated then $y(x) = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$.

(iii) If the roots $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ of the auxiliary equation are complex then $y(x) = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$.

a) Explain in two or more sentences which part of the above rules is justified by the reduction of order technique. Be as specific as you can about how the reduction of order calculation starts, and what it yields, but don't worry about the whole calculation.

b) Explain in two or more sentences which part of the above rules is justified by Euler's identity for complex numbers. In particular, write down Euler's identity.

Extra Credit. [2 point] Find a solution of the DE

$$ax^2y'' + bxy' + cy = 0$$

of the form $y = x^m$. (*Assume you are working on the interval* $(0, \infty)$, *where* x *is never zero.*) Show there are in fact two linearly-independent solutions of that form if $(b-a)^2 - 4ac > 0$.

TABLE 4.4.1 Trial Particular Solutions

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A \cos 4x + B \sin 4x$
6. $\cos 4x$	$A \cos 4x + B \sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x} \sin 4x$	$Ae^{3x} \cos 4x + Be^{3x} \sin 4x$
11. $5x^2 \sin 4x$	$(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$
12. $xe^{3x} \cos 4x$	$(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x$

Mass-spring-damper equivalent forms:

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt}$$

$$\iff \quad mx'' + \beta x' + kx = 0$$

$$\iff \quad x'' + 2\lambda x' + \omega^2 x = 0$$

where $\omega = \sqrt{\dfrac{k}{m}}, \quad \lambda = \dfrac{\beta}{2m}$

Converting from $x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$ to $x(t) = A \sin(\omega t + \phi)$:

$$A = \sqrt{c_1^2 + c_2^2}, \qquad \tan\phi = \frac{c_1}{c_2}$$