Name: SOLUTIONS

MATH F302 Differential Equations (Bueler)

SAMPLE Midterm Exam

No textbook or notes or calculator.

Please write your final answer in the box (if present).

1. (*10 pts*) Find the solution to the initial value problem:

[first-order linear]

$$\frac{dy}{dx} + 3y = x, \qquad y(0) = 5.$$

$$(e^{3x}y)' = x e^{3x}$$

$$e^{3x} y = \int x e^{3x} dx = x \frac{e^{3x}}{3} - \int 1 \cdot \frac{e^{3x}}{3} dx = \frac{1}{3} x e^{3x} - \frac{1}{3}\frac{1}{3} e^{3x} + c$$

$$y(x) = \frac{1}{3}x - \frac{1}{9} + c e^{-3x}$$

$$5 = y(0) = 0 - \frac{1}{9} + c \cdot 1$$

$$c = 5 + \frac{1}{9} = \frac{46}{9}$$

$$y(x) = \boxed{\frac{1}{3}x - \frac{1}{9} + \frac{46}{9} e^{-3x}}$$

2. (*10 pts*) Verify that the family of functions is a solution of the given differential equation:

$$\frac{dP}{dt} = P(1-P), \qquad P = \frac{c_1 e^t}{1 + c_1 e^t}$$

$$\frac{dP}{dt} = \frac{c_1 e^t (1 + \cancel{c_1 e^t}) - c_1 e^t (0 + \cancel{c_1 e^t})}{(1 + c_1 e^t)^2} = \frac{c_1 e^t}{(1 + c_1 e^t)^2}$$

$$P(1-P) = \frac{c_1 e^t}{1 + c_1 e^t}\left(\frac{1 + c_1 e^t}{1 + c_1 e^t} - \frac{c_1 e^t}{1 + c_1 e^t}\right)$$

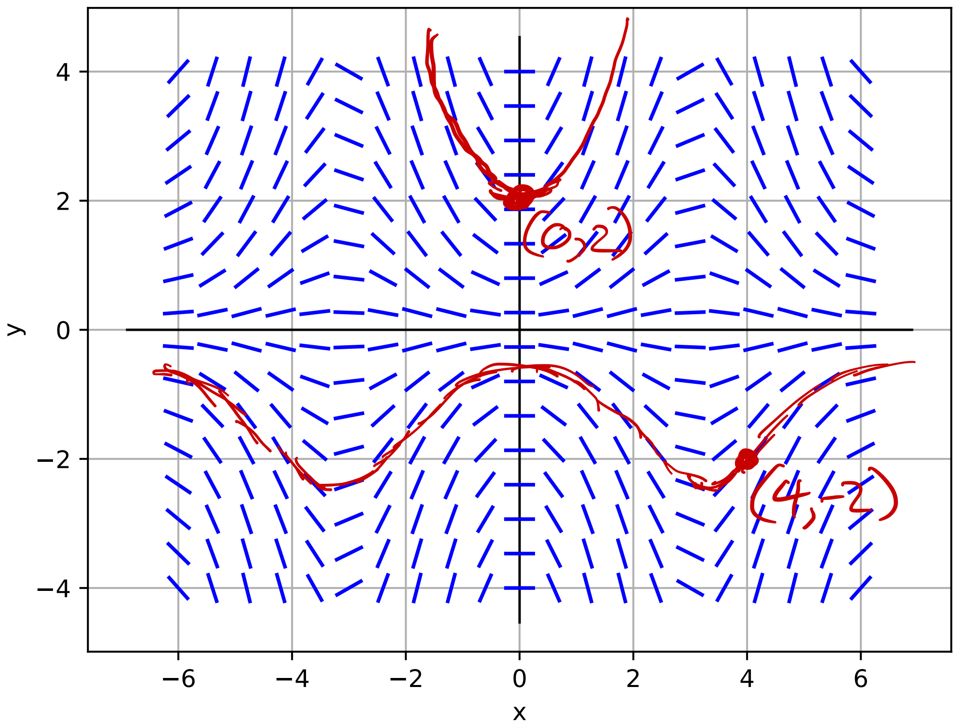
$$= \frac{c_1 e^t (1 + \cancel{c_1 e^t} - \cancel{c_1 e^t})}{(1 + c_1 e^t)} = \frac{c_1 e^t}{(1 + c_1 e^t)^2}$$

✓

3. (a) (*5 pts*) The ODE

$$\frac{dy}{dx} = y \sin x$$

has the direction field shown below. Sketch the solution which passes through $(x_0, y_0) = (4, -2)$. Please extend the sketched solution to the interval $-6 \le x \le 6$.



(b) (*10 pts*) Solve the following initial value problem—*give an exact formula $y(x)$ for the solution!*. Then sketch the solution you find on the direction field above:

[Separable]

$$\frac{dy}{dx} = y \sin x, \qquad y(0) = 2.$$

$$\frac{dy}{y} = \sin x \, dx$$

$$\ln|y| = -\cos x + C$$

$$y(x) = A e^{-\cos x}$$

$$2 = y(0) = A e^{-\cos 0} = A e^{-1}$$

$$A = +2e$$

$$y(x) = \boxed{2e \cdot e^{-\cos x}}$$

$$= 2e^{-\cos x + 1}$$

4. (*10 pts*) Find the general solution of the ODE:

[Separable]

$$\frac{dz}{dt} = e^{3t}e^{2z}$$

$$\frac{dz}{e^{2z}} = e^{3t}\,dt$$

$$\int e^{-2z}\,dz = \int e^{3t}\,dt$$

$$\frac{1}{-2}e^{-2z} = \frac{1}{3}e^{3t} + c$$

$$e^{-2z} = -\frac{2}{3}e^{3t} + c_1$$

$$-2z = \ln\left(c_1 - \frac{2}{3}e^{3t}\right)$$

$z(t) = \boxed{-\frac{1}{2}\ln\left(c_1 - \frac{2}{3}e^{3t}\right)}$

5. **(a)** (*10 pts*) Show that the following differential equation is exact:

$$(5y - 2x)y' - 2y = 0$$

$(5y-2x)\,dy - 2y\,dx = 0$

$\underbrace{-2y}_{M}\,dx + \underbrace{(5y-2x)}_{N}\,dy = 0$

$\dfrac{\partial M}{\partial y} = -2$, $\dfrac{\partial N}{\partial x} = -2$ $\therefore$ $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$\therefore$ exact

(b) (*10 pts*) Find the general solution to the differential equation in part **(a)**.

$M = -2y$, $N = 5y - 2x$

$\dfrac{\partial f}{\partial x} = M = -2y$

$f(x,y) = -2yx + h(y)$

$\therefore$ $5y - 2x = \dfrac{\partial f}{\partial y} = -2x + h'(y)$ $\therefore$ $h'(y) = 5y$

$\uparrow N$

$\therefore$ $h(y) = \frac{5}{2}y^2$

$\therefore$ $f(x,y) = -2yx + \frac{5}{2}y^2$

$$\boxed{-2xy + \frac{5}{2}y^2 = C}$$

6. **(a)** (*10 pts*) A drug is infused into a patient's bloodstream, and we denote the amount (grams) of the drug in the bloodstream by $x(t)$. The infusion is at a constant rate of r grams per second. Simultaneously, the drug is removed by the biochemical process in the patient's body at a rate proportional to the amount $x(t)$ of drug which is present at time t. Using k for the constant of proportionality, determine a differential equation (DE) for the amount $x(t)$.

$$\frac{dx(t)}{dt} = +r - k\,x(t)$$

(b) (*10 pts*) Assume there is no drug in the patient's bloodstream at time $t = 0$. Solve this initial value problem, which includes the DE in part **(a)**. (*Your solution will contain constants r and k.*)

[separable] $\dfrac{dx}{dt} = r - kx, \quad x(0) = 0$

$$\frac{dx}{r - kx} = dt$$

$$-\frac{1}{k}\ln|r - kx| = t + c$$

$$|r - kx| = e^{-kt + c_1}$$

$$r - kx = Ae^{-kt}$$

$$x(t) = \frac{r - Ae^{-kt}}{k}$$

$$0 = x(0) = \frac{r - A}{k}$$

$$\therefore A = r$$

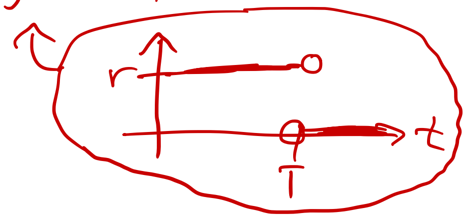
$$\therefore x(t) = \frac{r - re^{-kt}}{k}$$

$x(t) = \boxed{\dfrac{r}{k}\left(1 - e^{-kt}\right)}$

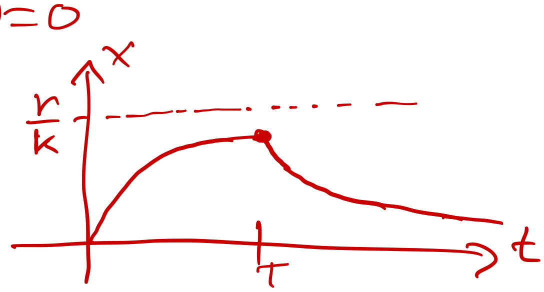
Extra Credit. (*3 pts*) A more realistic version of problem 6 would include the fact that a drug infusion does not go on forever. Assume it stops after time $T > 0$. Write down an ODE IVP for this improved model, and sketch a representative solution. (*The equation is solvable in pieces, but an exact solution is not needed here.*)

DE: $\dfrac{dx}{dt} = f(t) - kx, \quad x(0) = 0$

$$f(t) = \begin{cases} r, & 0 < t < T \\ 0, & t > T \end{cases}$$



sketch soln:



7. Consider the initial value problem: $y'' + y = 0, \quad y(0) = 2,\ y'(0) = 0$

(*10 pts*) Find the solution Show your steps.

$m^2 + 1 = 0$

$m = \pm i \quad \therefore \quad y(x) = c_1 \cos x + c_2 \sin x$

$2 = y(0) = c_1 \cdot 1 + c_2 \cdot 0 \qquad \therefore\ c_1 = 2,\ c_2 = 0$

$0 = y'(0) = -c_1 \cdot 0 + c_2 \cdot 1$

$y(x) =$ $\boxed{2 \cos x}$

8 (*10 pts*) Find the general solution: $y^{(4)} - 3y'' - 18y = 0$

(*Hint: Certain 4th-order polynomials can be factored if you know how to factor quadratics.*)

$m^4 - 3m^2 - 18 = 0$

$(m^2 - 6)(m^2 + 3) = 0$

$m = \pm\sqrt{6}$, $m = \pm\sqrt{3}\,i$

$y(x) = \boxed{c_1 e^{-\sqrt{6}x} + c_2 e^{+\sqrt{6}x} + c_3\cos(\sqrt{3}x)} + c_4\sin(\sqrt{3}x)$