Name: SOLUTIONS

MATH F302 UX1 Differential Equations (Bueler) 30 April or 1–2 May, 2019

Final Exam

Proctored. 150 minutes. 135 points total. No textbook or notes or calculator.
Please write your final answer $\boxed{\text{in the box}}$ if one is provided.

1. (*10 pts*) Find the general solution:

$$2y'' + 2y' + 5y = 0$$

$$2m^2 + 2m + 5 = 0$$

$$m = \frac{-2 \pm \sqrt{4-40}}{4} = -\frac{1}{2} \pm \frac{3}{2} i$$

$$y(x) = \boxed{c_1 e^{-\frac{1}{2}x} \cos\left(\tfrac{3}{2}x\right) + c_2 e^{-\frac{1}{2}x} \sin\left(\tfrac{3}{2}x\right)}$$

2. (*15 pts*) Find the simplified general solution. (*Hint.* Separation of variables. Partial fractions.)

$$\frac{dP}{dt} = P - P^2$$

$$\int \frac{dP}{P-P^2} = \int dt \longrightarrow \int \frac{1}{P} + \frac{1}{1-P}\, dP = t + c$$

$$\frac{1}{P(1-P)} = \frac{A}{P} + \frac{B}{1-P}$$

$$0P + 1 = A(1-P) + BP$$

$$A = 1$$

$$B - A = 0 \therefore B = 1$$

$$\ln|P| - \ln|1-P| = t + c$$

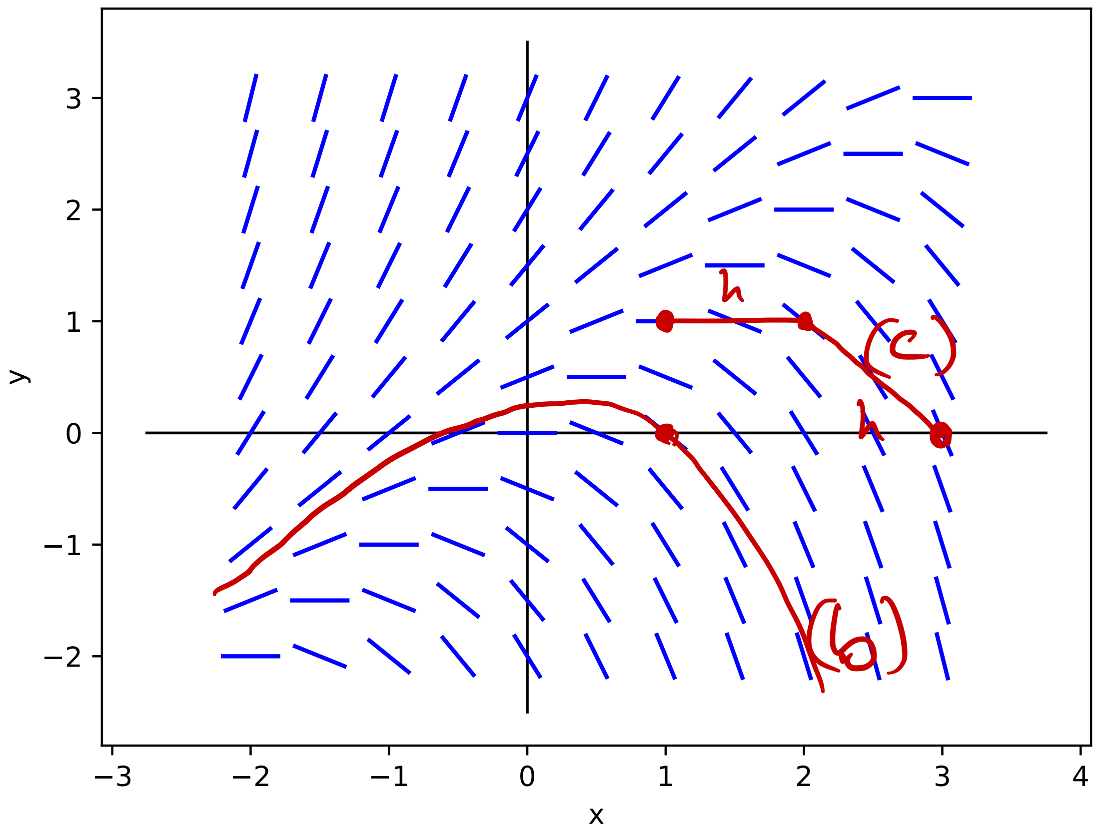
$$\ln\left|\frac{P}{1-P}\right| = t + c$$

$$\frac{P}{1-P} = Ae^t \longrightarrow P = Ae^t - Ae^t P$$

$$P(1 + Ae^t) = Ae^t$$

$$P(t) = \boxed{\frac{Ae^t}{1 + Ae^t}}$$

3. Consider this direction field, which is for a certain differential equation $\frac{dy}{dx} = f(x,y)$.



(a) (*10 pts*) Which differential equation is shown? Circle one.

A. $\frac{dy}{dx} = 2x - 1$

B. $\frac{dy}{dx} = y^2$

C. $\frac{dy}{dx} = x + y$

D. $\frac{dy}{dx} = y\sin(x)$

E. $\frac{dy}{dx} = y - x$

F. $\frac{dy}{dx} = xy$

[note $m=0$ on $y=x$]

(b) (*5 pts*) Sketch on the direction field the solution $y(x)$ to the initial value problem:

$$\frac{dy}{dx} = f(x,y), \qquad y(1) = 0.$$

Label this curve as **(b)**.

(c) (*5 pts*) Also sketch two steps of length $h = 1$ of the Euler method for solving this *different* initial value problem:

$$\frac{dy}{dx} = f(x,y), \qquad y(1) = 1.$$

Label this curve as **(c)**.

4. (*10 pts*) Solve this initial value problem for a first-order linear differential equation:

$$y' - y = -x, \qquad y(0) = 0$$

$$(e^{-x}y)' = -xe^{-x}$$

$$e^{-x}y(x) = -\int xe^{-x}\,dx = -\left(x(-e^{-x}) - \int(-e^{-x})dx\right)$$

$$= xe^{-x} - \int e^{-x}dx = xe^{-x} + e^{-x} + c$$

$$y(x) = x + 1 + ce^{x}$$

$$0 = y(0) = 0 + 1 + c \quad \therefore \quad c = -1$$

$y(x) =$ $\boxed{x + 1 - e^{x}}$

5. (*10 pts*) Use the definition of the Laplace transform to show that

$$\mathcal{L}\{y'(t)\} = sY(s) - y(0)$$

(Note we write $Y(s)$ for $\mathcal{L}\{y(t)\}$.)

$$\mathcal{L}\{y'(t)\} = \int_0^\infty e^{-st}y'(t)\,dt$$

$$= e^{-st}y(t)\Big]_0^\infty - \int_0^\infty(-se^{-st})y(t)\,dt$$

$$= 0 - y(0) + s\underbrace{\int_0^\infty e^{-st}y(t)\,dt}_{=Y(s)}$$

$$= sY(s) - y(0)$$

6. (*15 pts*) Use the Laplace transform to solve the initial value problem:

$y'' - y = 0, \qquad y(0) = 4, \quad y'(0) = 0$

$$s^2 Y(s) - s\underbrace{\cancel{y(0)}}_{4} - \underbrace{\cancel{y'(0)}}_{0} - Y(s) = 0$$

$$(s^2-1)\,Y(s) = 4s$$

$$Y(s) = \frac{4s}{(s+1)(s-1)} = \frac{A}{s+1} + \frac{B}{s-1}$$

$$4s + 0 = A(s-1) + B(s+1)$$

$$= (A+B)s + (B-A)$$

$$\left.\begin{aligned} A+B&=4 \\ B-A&=0 \end{aligned}\right\} \therefore \; 2A = 4$$

$$A = 2$$

$$B = 2$$

$$Y(s) = \frac{2}{s+1} + \frac{2}{s-1}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

check:

$y'' = +2e^{-t} + 2e^{t}$

$y'' - y = 0$ ✓

$y(0) = 4$ ✓

$y'(0) = -2\cdot 1 + 2\cdot 1 = 0$

$y(t) = \boxed{2e^{-t} + 2e^{t}}$

7. (*10 pts*) Verify that $y(t) = \frac{1}{2}\left(e^t - \sin(t) - \cos(t)\right)$ solves the initial value problem:

$$y'' + y = e^t, \qquad y(0) = 0, \quad y'(0) = 0$$

(Do not solve the problem from scratch. *Verify* that *all* parts of the IVP are satisfied.)

$$y'(t) = \frac{1}{2}(e^t - \cos(t) + \sin(t))$$

$$y''(t) = \frac{1}{2}(e^t + \sin(t) + \cos(t))$$

$$y(0) = \frac{1}{2}(e^0 - \sin(0) - \cos(0))$$

$$= \frac{1}{2}(1 - 0 - 1) = 0$$

$$y'(0) = \frac{1}{2}(1 - 1 + 0) = 0$$

$$y'' + y = \frac{1}{2}(e^t + \cancel{\sin(t)} + \cancel{\cos(t)}) + \frac{1}{2}(e^t - \cancel{\sin(t)} - \cancel{\cos(t)})$$

$$= \frac{1}{2}e^t + \frac{1}{2}e^t = e^t$$

8. (*10 pts*) Write the following ODE as a first-order system:

$y'' + y = e^t$

$w = y'$

$w' + y = e^t$

$$\boxed{\begin{aligned} y' &= w \\ w' &= -y + e^t \end{aligned}}$$

Extra Credit I. (*3 pts*) Write several lines of MATLAB/OCTAVE code to solve problem **7**, at the top of this page, using `ode45`. In particular, show how to generate the approximate value of $y(3)$.

```
>> f=@(t,z) [z(2); -z(1)+exp(t)];
>> [tt,zz] = ode45(f, [0,3],[0;0]);
>> zz(end,1)
```

9. (*15 pts*) Find and simplify the general solution of the system

$$\mathbf{X}' = \begin{pmatrix} -1 & 1 \\ 3 & 1 \end{pmatrix} \mathbf{X}$$

You may use the fact that the eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 3 & 1 \end{pmatrix}$ are

$$\lambda_1 = -2, \quad \lambda_2 = 2, \qquad \mathbf{K}_1 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$X_i(t) = K_i e^{\lambda_i t}$$

$$\therefore \quad X(t) = c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_2 t}$$

$$= c_1 \begin{pmatrix} 2 \\ -2 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{2t}$$

$$\mathbf{X}(t) = \begin{pmatrix} \boxed{2c_1 e^{-2t} + c_2 e^{2t}} \\ \boxed{-2c_1 e^{-2t} + 3c_2 e^{2t}} \end{pmatrix}$$

Extra Credit II. (*3 pts*) Compute and simplify $e^{\mathbf{A}t}$ if $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. (*Hint.* Pattern in $\mathbf{A}^k$?)

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ A^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \ A^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$$A^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ldots \quad \text{[repeats every 4]}$$

$$e^{At} = I + At + A^2 \frac{t^2}{2} + A^3 \frac{t^3}{3!} + A^4 \frac{t^4}{4!} + \cdots$$

$$= \left(I + A^2 \frac{t^2}{2} + A^4 \frac{t^4}{4!} + \cdots\right) + \left(At + A^3 \frac{t^3}{3!} + \cdots\right) = \begin{pmatrix} \cos t & 0 \\ 0 & \cos t \end{pmatrix} + \begin{pmatrix} 0 & \sin t \\ -\sin t & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

10. Consider the following initial value problem:

$$y'' - y = 0, \qquad y(0) = 0, \quad y'(0) = 2$$

(a) (*15 pts*) Solve the problem using power series, starting from the expression $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Your solution should make it clear how c_0 and c_1 are found, and it should include a recurrence relation for the coefficients. Find c_0, c_1, c_2, c_3, c_4 specifically.

$$y' = \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k, \quad y'' = \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k$$

$\underline{k = 0, 1, 2, \dots}$:

$$(k+2)(k+1) c_{k+2} - c_k = 0$$

recurrence relation

$$\left.\begin{aligned} 0 &= y(0) = c_0 \\ 2 &= y'(0) = c_1 \end{aligned}\right\} \text{found from initial conditions}$$

$\underline{k=0}$: $2 \cdot 1 \cdot c_2 - c_0 = 0 \quad \therefore c_2 = 0$

$\underline{k=1}$: $3 \cdot 2 \cdot c_3 - c_1 = 0 \quad \therefore c_1 = \frac{1}{3}$

$\underline{k=2}$: $4 \cdot 3 \cdot c_4 - c_2 = 0 \quad \therefore c_4 = 0$

$c_0 = \boxed{0}$

$c_1 = \boxed{2}$

$c_2 = \boxed{0}$

$c_3 = \boxed{1/3}$

$c_4 = \boxed{0}$

(b) (*5 pts*) What is the radius of convergence R of the power series in part **(a)**?

$R = \infty$ because ODE is constant-coeff.
($a_2(x)$ has <u>no</u> zeros)