Name: ______________________

MATH F302 UX1 Differential Equations (Bueler) — 30 April or 1–2 May, 2019

# Final Exam

**Proctored. 150 minutes. 135 points total. No textbook or notes or calculator.**
**Please write your final answer in the box if one is provided.**

**1.** (*10 pts*) Find the general solution:

$$2y'' + 2y' + 5y = 0$$

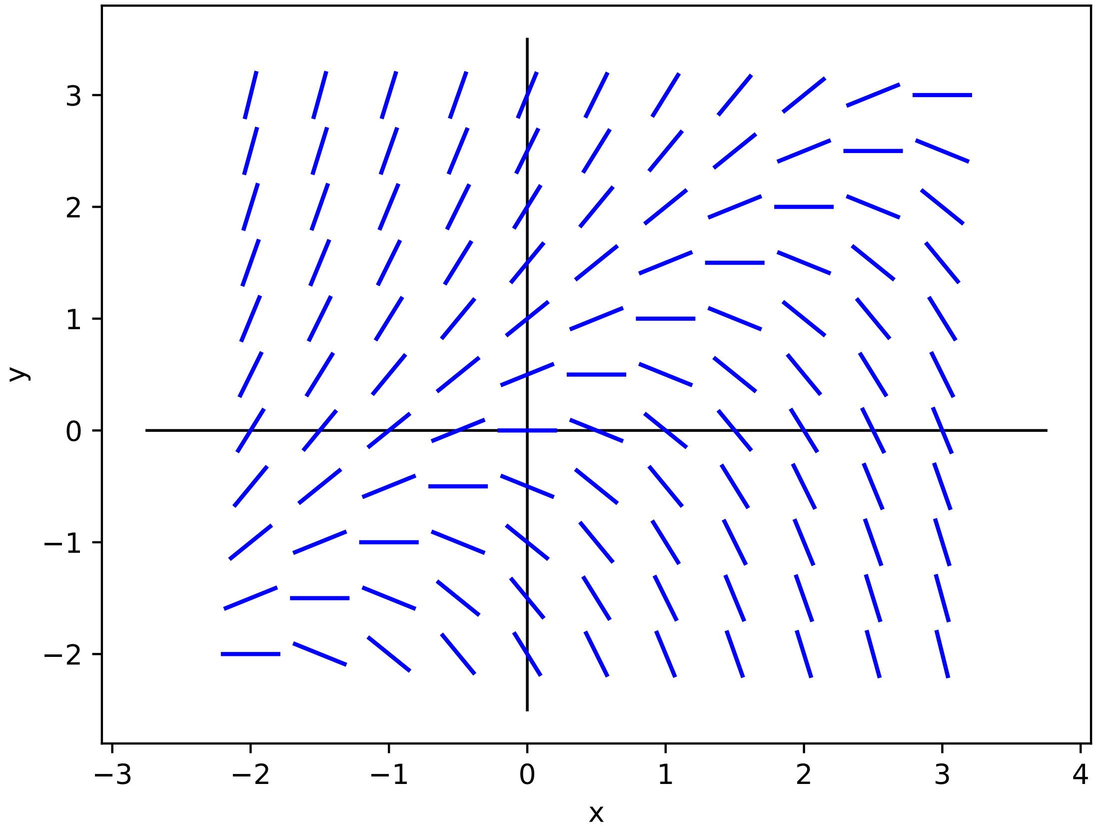
$y(x) =$ ☐

**2.** (*15 pts*) Find the simplified general solution. (*Hint.* Separation of variables. Partial fractions.)

$$\frac{dP}{dt} = P - P^2$$

$P(t) =$ ☐

**3.** Consider this direction field, which is for a certain differential equation $\frac{dy}{dx} = f(x, y)$.




**(a)** (*10 pts*) Which differential equation is shown? Circle one.

A. $\frac{dy}{dx} = 2x - 1$

B. $\frac{dy}{dx} = y^2$

C. $\frac{dy}{dx} = x + y$

D. $\frac{dy}{dx} = y\sin(x)$

E. $\frac{dy}{dx} = y - x$

F. $\frac{dy}{dx} = xy$

**(b)** (*5 pts*) Sketch on the direction field the solution $y(x)$ to the initial value problem:

$$\frac{dy}{dx} = f(x, y), \qquad y(1) = 0.$$

Label this curve as **(b)**.

**(c)** (*5 pts*) Also sketch two steps of length $h = 1$ of the Euler method for solving this *different* initial value problem:

$$\frac{dy}{dx} = f(x, y), \qquad y(1) = 1.$$

Label this curve as **(c)**.

**4.** (*10 pts*) Solve this initial value problem for a first-order linear differential equation:

$$y' - y = -x, \qquad y(0) = 0$$

$y(x) =$ ______

**5.** (*10 pts*) Use the definition of the Laplace transform to show that

$$\mathcal{L}\left\{y'(t)\right\} = sY(s) - y(0)$$

(Note we write $Y(s)$ for $\mathcal{L}\{y(t)\}$.)

**6.** (*15 pts*) Use the Laplace transform to solve the initial value problem:

$y'' - y = 0, \qquad y(0) = 4, \quad y'(0) = 0$

$y(t) =$ <br>

| |
|---|
| |

**7.** (*10 pts*) Verify that $y(t) = \frac{1}{2}\left(e^t - \sin(t) - \cos(t)\right)$ solves the initial value problem:

$$y'' + y = e^t, \qquad y(0) = 0, \quad y'(0) = 0$$

(Do not solve the problem from scratch. *Verify* that *all* parts of the IVP are satisfied.)

**8.** (*10 pts*) Write the following ODE as a first-order system:

$$y'' + y = e^t$$

| | |
|---|---|
| | = |
| | = |

**Extra Credit I.** (*3 pts*) Write several lines of MATLAB/OCTAVE code to solve problem **7**, at the top of this page, using `ode45`. In particular, show how to generate the approximate value of $y(3)$.

```
>>

>>

>>
```

**9.** (*15 pts*) Find and simplify the general solution of the system

$$\mathbf{X}' = \begin{pmatrix} -1 & 1 \\ 3 & 1 \end{pmatrix} \mathbf{X}$$

You may use the fact that the eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 3 & 1 \end{pmatrix}$ are

$$\lambda_1 = -2, \quad \lambda_2 = 2, \qquad \mathbf{K}_1 = \begin{pmatrix} 2 \\ -2 \end{pmatrix}, \quad \mathbf{K}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\mathbf{X}(t) = \begin{pmatrix} \boxed{\phantom{xxxxxxxxxx}} \\ \boxed{\phantom{xxxxxxxxxx}} \end{pmatrix}$$

**Extra Credit II.** (*3 pts*) Compute and simplify $e^{\mathbf{A}t}$ if $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. (*Hint.* Pattern in $\mathbf{A}^k$?)

**10.** Consider the following initial value problem:

$$y'' - y = 0, \qquad y(0) = 0, \quad y'(0) = 2$$

**(a)** (*15 pts*) Solve the problem using power series, starting from the expression $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Your solution should make it clear how $c_0$ and $c_1$ are found, and it should include a recurrence relation for the coefficients. Find $c_0, c_1, c_2, c_3, c_4$ specifically.

$c_0 =$ ☐

$c_1 =$ ☐

$c_2 =$ ☐

$c_3 =$ ☐

$c_4 =$ ☐

**(b)** (*5 pts*) What is the radius of convergence $R$ of the power series in part **(a)**?

Brief Table of Integrals

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c,\ \ n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\int u\ dv = uv - \int v du$$

$$\int e^x\,dx = e^x + c$$

$$\int a^x\,dx = \frac{1}{\ln a}a^x + c$$

$$\int \ln x\,dx = x\ln x - x + c$$

$$\int \sin x\,dx = -\cos x + c$$

$$\int \cos x\,dx = \sin x + c$$

$$\int \tan x\,dx = \ln|\sec x| + c$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + c$$

$$\int \sec^2 x\,dx = \tan x + c$$

$$\int \sec x\tan x\,dx = \sec x + c$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\ln\left|\frac{x+a}{x-a}\right| + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\,\mathrm{arcsec}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left|x+\sqrt{x^2-a^2}\right| + c$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left|x+\sqrt{x^2+a^2}\right| + c$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + c$$

$$\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + c$$

| **Maclaurin Series** | **Interval** |
|---|---|
| $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$ | $(-\infty,\infty)$ |
| $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$ | $(-\infty,\infty)$ |
| $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$ | $(-\infty,\infty)$ |
| $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}$ | $[-1,1]$ (2) |
| $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n}$ | $(-\infty,\infty)$ |
| $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n+1}$ | $(-\infty,\infty)$ |
| $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}x^n$ | $(-1,1]$ |
| $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$ | $(-1,1)$ |

**TABLE 4.4.1** Trial Particular Solutions

| $g(x)$ | Form of $y_p$ |
|---|---|
| **1.** 1 (any constant) | $A$ |
| **2.** $5x + 7$ | $Ax + B$ |
| **3.** $3x^2 - 2$ | $Ax^2 + Bx + C$ |
| **4.** $x^3 - x + 1$ | $Ax^3 + Bx^2 + Cx + E$ |
| **5.** $\sin 4x$ | $A\cos 4x + B\sin 4x$ |
| **6.** $\cos 4x$ | $A\cos 4x + B\sin 4x$ |
| **7.** $e^{5x}$ | $Ae^{5x}$ |
| **8.** $(9x-2)e^{5x}$ | $(Ax + B)e^{5x}$ |
| **9.** $x^2e^{5x}$ | $(Ax^2 + Bx + C)e^{5x}$ |
| **10.** $e^{3x}\sin 4x$ | $Ae^{3x}\cos 4x + Be^{3x}\sin 4x$ |
| **11.** $5x^2\sin 4x$ | $(Ax^2 + Bx + C)\cos 4x + (Ex^2 + Fx + G)\sin 4x$ |
| **12.** $xe^{3x}\cos 4x$ | $(Ax + B)e^{3x}\cos 4x + (Cx + E)e^{3x}\sin 4x$ |

Table of Laplace Transforms:

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{t\} = \frac{1}{s^2}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\left\{t^{-1/2}\right\} = \frac{\sqrt{\pi}}{s^{1/2}}$$

$$\mathcal{L}\left\{t^{1/2}\right\} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

$$\mathcal{L}\{t^\alpha\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$$

$$\mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}$$

$$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}$$

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$$

$$\mathcal{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2}$$

$$\mathcal{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2}$$

$$\mathcal{L}\left\{te^{at}\right\} = \frac{1}{(s-a)^2}$$

$$\mathcal{L}\left\{t^n e^{at}\right\} = \frac{n!}{(s-a)^{n+1}}$$

$$\mathcal{L}\left\{e^{at}\sin(kt)\right\} = \frac{k}{(s-a)^2+k^2}$$

$$\mathcal{L}\left\{e^{at}\cos(kt)\right\} = \frac{s-a}{(s-a)^2+k^2}$$

$$\mathcal{L}\{t\sin(kt)\} = \frac{2ks}{(s^2+k^2)^2}$$

$$\mathcal{L}\{t\cos(kt)\} = \frac{s^2-k^2}{(s^2+k^2)^2}$$

$$\mathcal{L}\left\{e^{at}f(t)\right\} = F(s-a)$$

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \frac{e^{-as}}{s}$$

$$\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\} = e^{-as}F(s)$$

$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}F(s)$$

$$\mathcal{L}\left\{f^{(n)}(t)\right\} = s^n F(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$$

$$(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

$$\mathcal{L}\{f*g\} = F(s)G(s)$$

space below is available for computations clearly label anything you want graded

SPACE BELOW IS AVAILABLE FOR COMPUTATIONS CLEARLY LABEL ANYTHING YOU WANT GRADED