Name: SOLUTIONS

Math 302 Differential Equations (Bueler) Wednesday 18 October 2023

Midterm Exam

In-class. No book, notes, electronics, calculator, internet access, or communication with other people. Please write your solution in the box if provided. 100 points possible. 65 minutes maximum!

1. (*6 pts*) Solve the differential equation by separation of variables: $\frac{dy}{dx} - 2xy^2 = 0$

$$\int \frac{dy}{y^2} = \int 2x\,dx = x^2 + c$$

$$-y^{-1} = x^2 + c$$

$$\boxed{y(x) = \frac{-1}{x^2+c}}$$

2. (*6 pts*) Verify that $y(t) = 1/(c-t)$ is a one-parameter family of solutions of the differential equation $\frac{dy}{dt} = y^2$.

$$y(t) = (c-t)^{-1}$$

$$\frac{dy}{dt} = -(c-t)^{-2} \cdot (-1) = +(c-t)^{-2}$$

$$y^2 = [(c-t)^{-1}]^2 = (c-t)^{-2}$$

// ✓ verified

3. (*15 pts*) Determine whether the differential equation is exact. If it is exact, solve it and write the solution in the box: $(8y^3 - 4x)\frac{dy}{dx} = 5x + 4y$

$$\underbrace{(-5x-4y)}_{M}dx + \underbrace{(8y^3-4x)}_{N}dy = 0$$

exact? $\frac{\partial M}{\partial y} = -4 = \frac{\partial N}{\partial x}$ ✓ yea

so: $f(x,y) = \int M\,dx = -\frac{5}{2}x^2 - 4yx + g(y)$

$$8y^3 \cancel{-4x} = N = \frac{\partial f}{\partial y} = \cancel{-4x} + g'(y)$$

$$g'(y) = 8y^3$$

$$g(y) = 2y^4$$

$$f(x,y) = -\frac{5}{2}x^2 - 4yx + 2y^4$$

needs to be equation in y & x (not "f=")

$$\boxed{-\frac{5}{2}x^2 - 4yx + 2y^4 = c}$$

4. (*15 pts*) Solve the initial value problem, assuming L, R, E, i_0 are constants:

$$L\frac{di}{dt} + Ri = E, \quad i(0) = i_0$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$$

as linear DE:

you can also solve as separable

$$\mu = e^{\int (R/L)\,dt} = e^{(R/L)t}$$

$$\left(e^{(R/L)t}\, i(t)\right)' = \frac{E}{L} e^{(R/L)t}$$

$$e^{(R/L)t}\, i(t) = \frac{E}{L}\int e^{(R/L)t}\,dt$$

$$= \frac{E}{L}\left(\frac{L}{R}e^{(R/L)t} + c\right)$$

$$= \frac{E}{R}e^{(R/L)t} + \tilde{c}$$

$$i(t) = \frac{E}{R} + \tilde{c}e^{-(R/L)t}$$

$$i_0 = i(0) = \frac{E}{R} + \tilde{c} \quad \therefore \quad \tilde{c} = i_0 - \frac{E}{R}$$

$$\boxed{i(t) = \frac{E}{R} + \left(i_0 - \frac{E}{R}\right)e^{-(R/L)t}}$$

5. **(a)** (*8 pts*) Find the general solution: $y'' - y' - 12y = 0$

$m^2 - m - 12 = 0$

$(m-4)(m+3) = 0$

$$\boxed{y(x) = c_1 e^{4x} + c_2 e^{-3x}}$$

(b) (*8 pts*) Show that your general solution in part **(a)** is built from a fundamental set of solutions.

$$W(e^{4x}, e^{-3x}) = \det \begin{bmatrix} e^{4x} & e^{-3x} \\ 4e^{4x} & -3e^{-3x} \end{bmatrix}$$

$$= -3e^{x} - 4e^{x} = -7e^{x} \neq 0$$

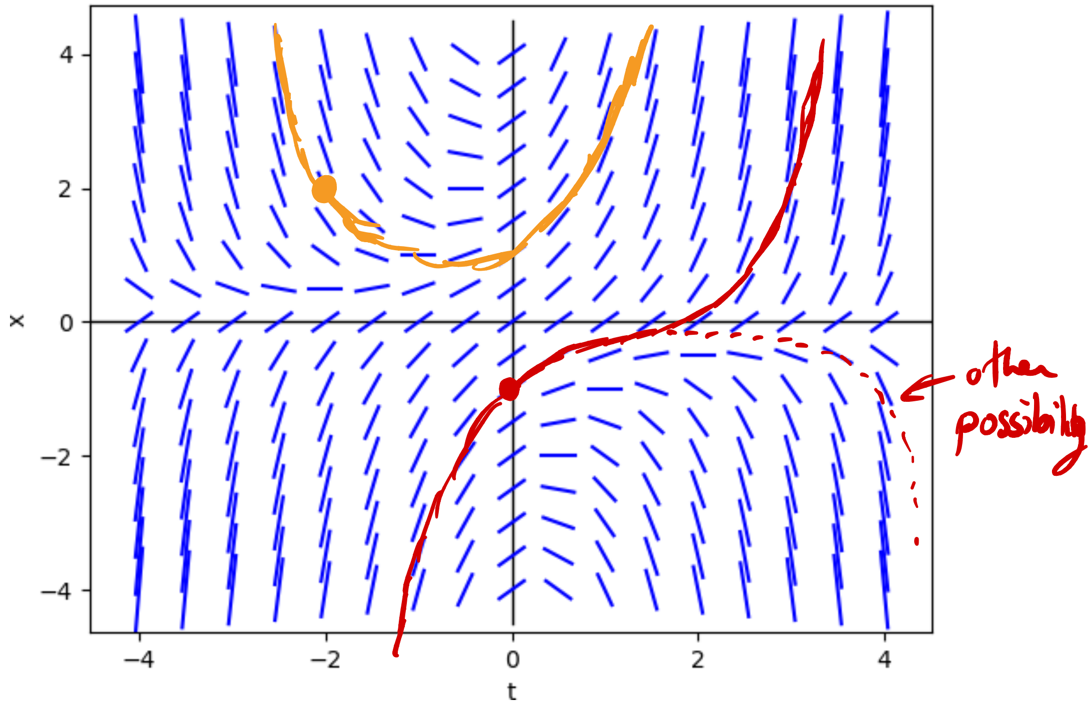
for any x

so $\{e^{4x}, e^{-3x}\}$ is a linearly-independent set of 2 functions, so a fundamental set

6. **(a)** (*5 pts*) The direction field of the following differential equation is shown: $\dfrac{dx}{dt} = 1 + tx$

For each of the following points, sketch an approximate solution curve.

- $x(0) = -1$
- $x(-2) = 2$



(b) (*8 pts*) Find the general solution of the differential equation in part **(a)**. You may write the solution in integral form if you do not know how to do an integral.

as linear:

$$\frac{dx}{dt} - tx = 1$$

$$\mu = e^{\int -t\,dt} = e^{-t^2/2}$$

$$\left(e^{-t^2/2} x(t)\right)' = e^{-t^2/2}$$

$$e^{-t^2/2} x(t) = \int e^{-t^2/2}\,dt$$

$x(t) =$ $e^{t^2/2}\left(\int e^{-t^2/2}dt + c\right)$

optional, because "$\int$" already has unknown constant also fine: $\boxed{\frac{dT}{dt} = k(T_m - T)}$ (with $k>0$)

7. (a) (*5 pts*) Newton's law of cooling/warming says that the rate of change of an object's temperature is proportional to the difference between its temperature and a constant ambient temperature. Write this differential equation, a model for the temperature $T(t)$. (*Hint. This model has two constants.*)

$$\boxed{\frac{dT}{dt} = k(T - T_m)}$$

note $k<0$ because heat flows from hot to cold

(b) (*5 pts*) Find the general solution of the differential equation in part **(a)**.

separable: $\displaystyle\int \frac{dT}{T-T_m} = \int k\,dt = kt + C$

$$\ln|T - T_m| = kt + C$$

$$|T - T_m| = e^C e^{kt}$$

$$T - T_m = Ae^{kt}$$

$$\boxed{T(t) = T_m + Ae^{kt}}$$

(c) (*5 pts*) A thermometer—*it is the object here*—is initially at room temperature 20° C. It is put in boiling water at 100° C. After 2 seconds the thermometer reads 60° C. What is the constant of proportionality in the above model?

$$\left.\begin{aligned} T(0) &= 20 \\ T_m &= 100 \end{aligned}\right\} \quad 20 = 100 + A \therefore A = -80$$

$$\therefore T(t) = 100 - 80e^{kt}$$

$$T(2) = 60$$

$$60 = 100 - 80e^{2k}$$

$$\frac{1}{2} = \frac{-40}{-80} = e^{2k}$$

$$\boxed{k = -\ln(2)/2}$$

this is fine too

$$k = \frac{\ln(1/2)}{2} = -\ln(2)/2$$

$k = \ln 2/2$ if "$\frac{dT}{dt} = k(T_m - T)$" above

8. **(a)** (*7 pts*) Find the general solution of this third-order, constant-coefficient, homogeneous, and linear differential equation: $y''' + y' = 0$

$m^3 + m = 0$

$m(m^2+1) = 0$

$m = 0, \quad m = \pm i$

$m=0 \to e^0 = 1$, $\quad m=\pm i \to \cos x, \sin x$

$$\boxed{y(x) = c_1 + c_2\cos x + c_3 \sin x}$$

(b) (*7 pts*) The general solution in part **(a)** is a linear combination of three specific functions $f_1(x), f_2(x), f_3(x)$. State these functions and then compute and simplify the Wronskian.

$f_1(x) = 1, \quad f_2(x) = \cos x, \quad f_3(x) = \sin x$

$$W(f_1, f_2, f_3) = \det\begin{bmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{bmatrix} = 1\cdot(\sin^2 x + \cos^2 x) - 0 + 0 = 1$$

$$\boxed{W(f_1, f_2, f_3) = 1}$$

Extra Credit. (*3 pts*) Find the general solution on $(0, \infty)$: $xy'' - y' = 0$

$w = y'$: $\quad xw' - w = 0$

$w' - \frac{1}{x} w = 0$

no abs. val. needed

$$\mu = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln(1/x)} = \frac{1}{x}$$

$$\left(\frac{1}{x} w(x)\right)' = 0$$

$y'(x) = w(x) = Cx \longrightarrow y(x) = c_1\frac{x^2}{2} + c_2$