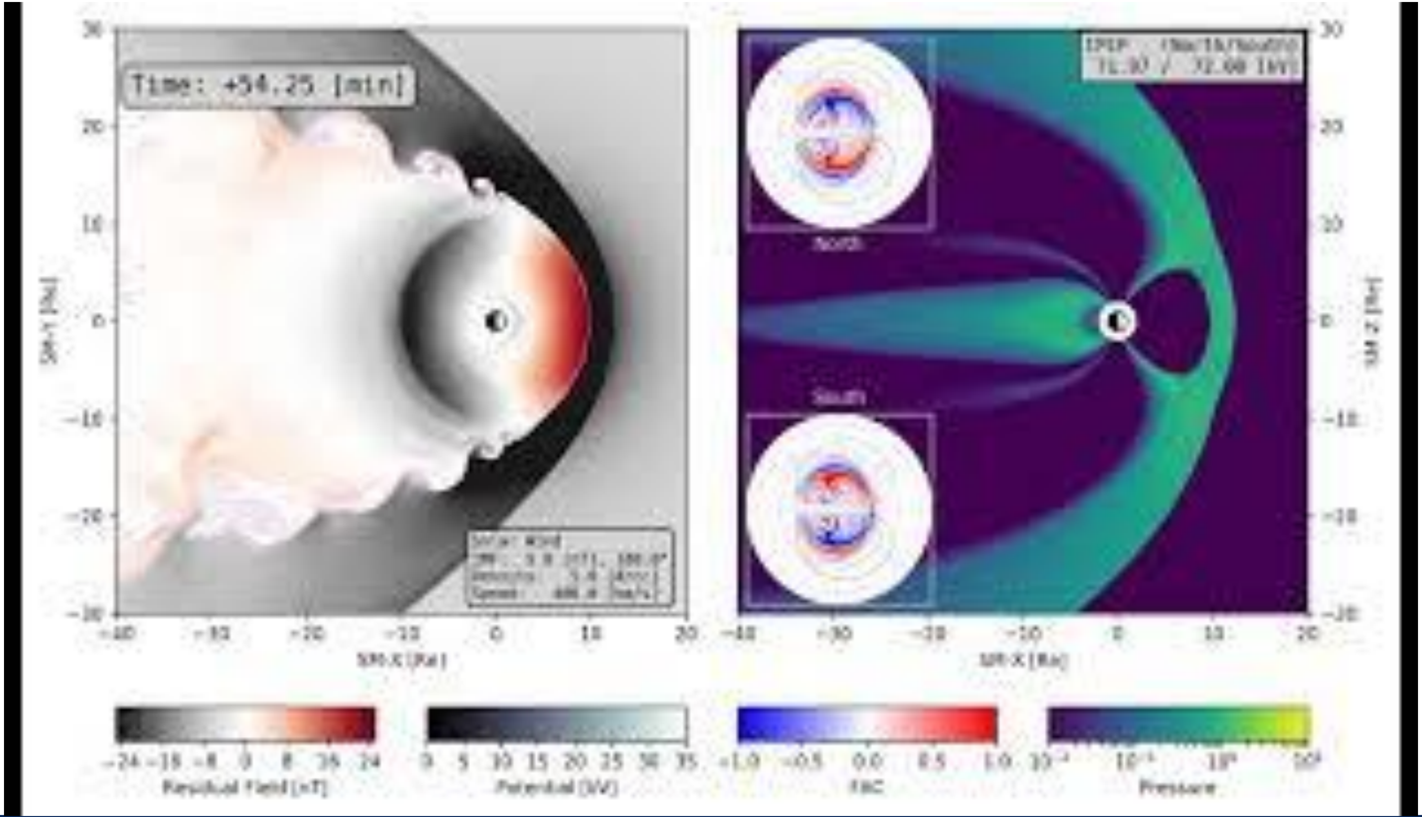
Introduction to Magnetohydrodynamics



Background

- Magnetohydrodynamics describes bulk motion of a charge fluids
- Derivable from the moments of the particle distribution function
- Valid in a wide range of physics processes
  - Low frequencies
  - Low ion-electrons collisions

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  - Hall, or two-fluid
  - Ressitive
  - Ideal
    - Flux conservative

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  - Hall, or two-fluid
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  - Ideal
    - Flux conservative
- Ions = positive
  - Dusty plasma have large negative charge carriers
  - Generally not valid for MHD

Maxwell's Equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Faraday's Law

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$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

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No magnetic monopoles

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$$\nabla \cdot \mathbf{E} = \frac{\rho_{\text{charge}}}{\epsilon_0}$$

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$$\nabla \cdot \mathbf{E} = \frac{\rho_{\text{charge}}}{\epsilon_0}$$

Gauss's Law

Where, $\rho_{\text{charge}} = q_e n_e + \sum_i -q_e Z_i n_i$

For space/astrophysical processes, we assume quasi-neutrality: $\rho_{\text{charge}} = 0$

Debye Length

Quasi-neutrality is valid for processes with scale lengths larger than the Debye length: $\lambda_{\mathrm{D}} = \sqrt{\dfrac{\epsilon_0 k_B / q_e^2}{n_e/T_e + \boxed{\sum_i Z_i^2 n_i / T_j}}}$

Ion term is often dropped, since ions often have low mobility compared to the process timescale: $\lambda_{\mathrm{D}} = \sqrt{\dfrac{\epsilon_0 k_B T_e}{n_e q_e^2}}$

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Tokamak	Ionosphere	Magnetosphere	1 AU Solar Wind
10^{-4} m	10^{-3} m	~10^{2} m	10 m

Maxwell's Equations

- Two (equally valid) non-relativistic limits
  - $E \ll cB$ - Magnetically dominated
  - $E \gg cB$ - Electrically dominated

Maxwell's Equations

- Two (equally valid) non-relativistic limits
  - E << cB - Magnetically dominated
  - E >> cB - Electrically dominated
- Quasi-neutral, Magnetically dominated Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathrm{B}}{\partial t}$$

$$\nabla \times \mathrm{B} = \mu_0 \mathbf{J}$$

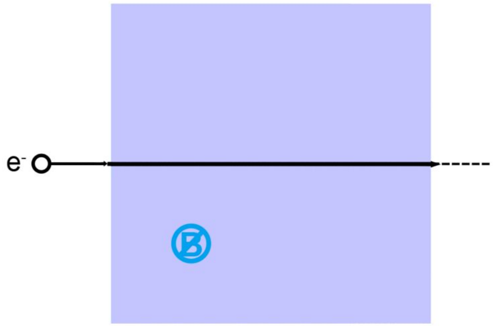
$$\nabla \cdot \mathrm{B} = 0$$

$$\nabla \cdot \mathbf{E} = 0$$

Lorentz Force and Ohm's Law

- The Lorentz force describes the interaction between charges and electromagnetic fields

$$F = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$



Lorentz Force and Ohm’s Law

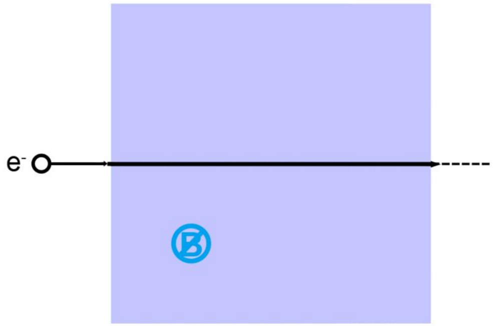
- The Lorentz force describes the interaction between charges and electromagnetic fields

$$F = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

- Ohm’s law describes the the relation between current and Electric field
  - Induced currents from the Lorentz force and external magnetic fields also must be considered when moving,

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

  - Where σ us the electrical conductivity



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Generalized Ohm's Law

- Including all potentially relevant terms, Ohm's law takes the form:

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B} + \frac{1}{nq}\mathbf{J} \times \mathbf{B} - \frac{1}{nq}\boldsymbol{\nabla} p_e + \eta \mathbf{J} + \frac{m_e}{nq^2}\left[\frac{\partial \mathbf{J}}{\partial t} + \boldsymbol{\nabla} \cdot (\mathbf{J}\mathbf{u} + \mathbf{u}\mathbf{J})\right]$$

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Motional
electric field

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- Assuming high conductivity, and negligible pressure, inertial, and Hall terms, the Ohm's law we will use is just the motional electric field:

$$\mathbf{E} = -\mathbf{u} \times \mathbf{B}$$

Frozen-in condition

- Magnetic Reynolds number is, $R_{\mathrm{m}} = \dfrac{\mu_0 \nabla \times (\mathbf{u} \times \mathbf{B}}{\eta \nabla^2 \mathbf{B}}$

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- This allows us to eliminate **E** in Faraday's law with the motional electric field from Ohm's law:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})$$

Continuity Equation

- Assuming plasma consists of electrons and one positive ion species
  - Starting with two fluid equations for mass and momentum
  - Since we are assuming quasi-neutrality: $n_i \approx n_e$

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- Total mass density: $\rho = m_i n_i + m_e n_e$
- Center of mass velocity: $\rho \mathbf{u} = n_e m_e \mathbf{u}_e + n_i m_i \mathbf{u}_i$

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- Center of mass velocity: $\rho \mathbf{u} = n_e m_e \mathbf{u}_e + n_i m_i \mathbf{u}_i$
- Since $m_e/m_i = 5.45\times10^{-4} << 1$, we can approximate:

$$\mathbf{u} = \frac{n_e m_e \mathbf{u}_e + n_i m_i \mathbf{u}_i}{n_e m_e + n_i m_i} \approx \mathbf{u}_i$$

$$\rho = n_e m_e + n_i m_i \approx n_e m_i$$

Continuity Equation

- Two fluid equations continuity equations:

$$m_i \frac{\partial n_i}{\partial t} + m_i \boldsymbol{\nabla} \cdot n_i \mathbf{u}_i = 0$$

$$m_e \frac{\partial n_e}{\partial t} + m_e \boldsymbol{\nabla} \cdot n_e \mathbf{u}_e = 0$$

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- Add them together:

$$m_i \frac{\partial n_i}{\partial t} + m_e \frac{\partial n_e}{\partial t} + m_i \boldsymbol{\nabla} \cdot n_i \mathbf{u}_i + m_e \boldsymbol{\nabla} \cdot n_e \mathbf{u}_e = 0$$

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- Rearranging terms:

$$\frac{\partial(m_i n_i + m_e n_e)}{\partial t} + \boldsymbol{\nabla}\cdot(n_i m_i\mathbf{u}_i + n_e m_e\mathbf{u}_e) = 0$$

$$\frac{\partial(n_i m_i + n_e m_e)}{\partial t} + \boldsymbol{\nabla}\cdot\left[(n_i m_i + n_e m_e)\frac{n_i m_i\mathbf{u}_i + n_e m_e\mathbf{u}_e}{n_i m_i + n_e m_e}\right] = 0$$

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- Substitute **u** and ρ:

$$\mathbf{u} = \frac{n_i m_i \mathbf{u}_i + n_e m_e \mathbf{u}_e}{n_i m_i + n_e m_e}$$

$$\rho = m_i n_i + m_e n_e$$

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

Momentum Equation

- Only considering the Lorentz force for two fluids:

$$m_i n_i \frac{d\mathbf{u}_i}{dt} = q_i n_i (\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) \qquad m_e n_e \frac{d\mathbf{u}_e}{dt} = q_e n_e (\mathbf{E} + \mathbf{u}_e \times \mathbf{B})$$

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- Add them together:

$$\frac{d}{dt}(n_i m_i \mathbf{u}_i + n_e m_e \mathbf{u}_e) = nq(\mathbf{u}_i - \mathbf{u}_e) \times \mathbf{B}$$

- **E** is eliminated since, $q_i = -q_e = |q|$
- Also, $n = n_i \approx n_e$

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Momentum Equation

- Now we have one equation: $\frac{d}{dt}(n_i m_i \mathbf{u}_i + n_e m_e \mathbf{u}_e) = nq(\mathbf{u}_i - \mathbf{u}_e) \times \mathbf{B}$

- Current density is defined as

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- Substituting the current density and rearranging:

$$n(m_i + m_e)\frac{d}{dt}\left[\frac{n(m_i \mathbf{u}_i + m_e \mathbf{u}_e)}{n(m_i + m_e)}\right] = \mathbf{J} \times \mathbf{B}$$

$$(n_i m_i + n_e m_e)\frac{d}{dt}\left[\frac{n_i m_i \mathbf{u}_i + n_e m_e \mathbf{u}_e}{n_i m_i + n_e m_e}\right] = \mathbf{J} \times \mathbf{B}$$

Momentum Equation

- From previous step: $$(n_i m_i + n_e m_e)\frac{d}{dt}\left[\frac{n_i m_i \mathbf{u}_i + n_e m_e \mathbf{u}_e}{n_i m_i + n_e m_e}\right] = \mathbf{J} \times \mathbf{B}$$

- Similar to the continuity equation, substitute **u** and ρ,

$$\mathbf{u} = \frac{n_i m_i \mathbf{u}_i + n_e m_e \mathbf{u}_e}{n_i m_i + n_e m_e}$$

$$\rho = m_i n_i + m_e n_e$$

$$\rho\frac{d\mathbf{u}}{dt} = \mathbf{J} \times \mathbf{B}$$

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$$\rho \frac{d\mathbf{u}}{dt} = \mathbf{J} \times \mathbf{B}$$

- Advective derivative gives a more familiar form,

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{J} \times \mathbf{B}$$

Momentum Equation

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{J} \times \mathbf{B}$$

- Substitute **J** with Ampère's law, $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B}$$

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- Apply vector identities[1]:

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u}\cdot\nabla)\mathbf{u} = \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} - \frac{\nabla B^2}{2\mu_0}$$

1: Specifically, $\nabla(\mathrm{A}\cdot\mathrm{B}) = \mathrm{A}\times(\nabla\times\mathrm{B}) + \mathrm{B}\times(\nabla\times\mathrm{A}) + (\mathrm{A}\cdot\nabla)\mathrm{B} + (\mathrm{B}\cdot\nabla)\mathrm{A}$

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Magnetic pressure

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- We just considered the Lorentz force, but many other forces and momentum sources/sinks may be present
  - Plasma pressure
  - Gravity
  - Collisions

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$$p = nk_BT \qquad \text{or} \qquad p = R\rho T$$

- Takes the form of

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0 \qquad \text{or} \qquad \boxed{\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma}$$

- Where, $\gamma = \frac{C_p}{C_v} = 5/3$, is the ratio between heat capacity at constant pressure and heat capacity at constant volume

Ideal MHD Equations

Continuity:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0$$

Momentum:

$$\rho \frac{\partial \mathbf{u}}{\partial t} + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{1}{\mu_0}(\mathbf{B} \cdot \nabla)\mathbf{B} - \frac{\nabla B^2}{2\mu_0} + \nabla p$$

Ideal induction:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B})$$

Equation of State:

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^{\gamma}$$

Resistive MHD Energy Equation

$$\frac{\partial}{\partial t}\underbrace{\left(\frac{1}{\gamma-1}p+\frac{1}{2}\rho u^2+\frac{B^2}{2\mu_0}\right)}_{\text{Total Energy}} = -\nabla\cdot\left[\left(\frac{1}{\gamma-1}p+\frac{1}{2}\rho u^2+\frac{B^2}{2\mu_0}\right)\mathbf{u}-\frac{1}{\mu_0}(\mathbf{u}\cdot\mathbf{B})\mathbf{B}+\frac{\eta}{\mu_0}(\mathbf{J}\times\mathbf{B})\right]$$

Internal Energy | Kinetic Energy | Magnetic Energy | Source/Sinks of Electromagnetic Energy

Ideal MHD + Energy Equation + Source