Fluids & Solids graduate seminar

23 Jan. 2025

(MATH 692, 1.0 credit, crn 35130)

- credit <u>not</u> required!
- if you want 1.0 credit, expect to give one talk ... please volunteer!

continuum mechanics through Navier-Stokes

plan:

- notation & product rules } just calc 3
- divergence theorem & integration by parts
- general (integral) conservation ... and its derivatives form } probably new to math grads?
- conservation of mass } actual physics
- " " momentum
- Navier-Stokes for incompressible fluids } a specific model

"general conservation law"

<u>def.</u> suppose $s : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$, i.e. $s(t,x)$, is a scalar <u>source</u> function, ~~and~~ RECALL

$\vec{F} : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$, i.e. $\vec{F}(t,x)$, is a vector-valued <u>flux</u> function. We say the quantity of which φ is the density $\cancel{\varphi(t,x)}$ is conserved,

where $\varphi : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is the scalar <u>density</u>, if

$$\boxed{\frac{d}{dt}\left(\int_{\Omega} \varphi \, dx\right) = -\int_{\partial\Omega} \vec{F} \cdot \hat{n} \, dS + \int_{\Omega} s \, dx}$$

for all $\Omega \subset \mathbb{R}^3$

"φ is conserved" means we know how the amount of φ in Ω, namely

RECALL

$$\int_\Omega \varphi \, dx$$

changes in time, based on knowing how much leaves through the boundary $\left(-\int_{\partial\Omega} \vec{F} \cdot \hat{n} \, dS\right)$ and how much is created inside $\left(\int_\Omega s \, dx\right)$

derivative form of (general) conservation

RECALL

- true over every $\Omega \subset \mathbb{R}^3$:

$$\frac{d}{dt}\left(\int_{\Omega} \varphi \, dx\right) \overset{\circledast}{=} -\int_{\partial\Omega} \vec{F}\cdot\hat{n}\, ds + \int_{\Omega} s\, dx$$

- time derivative:

$$\frac{d}{dt}\left(\int_{\Omega} \varphi \, dx\right) \overset{\text{LDC}}{=} \int_{\Omega} \frac{\partial \varphi}{\partial t}\, dx$$

- apply divergence theorem to flux surface integral:

$$\int_{\partial\Omega} \vec{F}\cdot\hat{n}\, ds \overset{\text{DT}}{=} \int_{\Omega} \nabla\cdot\vec{F}\, dx$$

RECALL

- thus rewrite ⊛ as

$$\int_{\Omega} \left(\frac{\partial \varphi}{\partial t} + \nabla \cdot \vec{F} - S \right) dx = 0$$

- since this is true for every $\Omega \subset \mathbb{R}^3$, we get a partial differential equation (PDE) form of conservation:

$$\boxed{\frac{\partial \varphi}{\partial t} + \nabla \cdot \vec{F} = S}$$

in real analysis:

$$\int f \chi_S \, dx = 0 \quad \forall S$$

$$\Rightarrow$$

$$f = 0 \text{ a.e.}$$

The Eulerian view

- in this (general) conservation view, the "control volume" $\Omega \subset \mathbb{R}^3$ is fixed and does not move
- this view is also taken when you fix your coordinates (or numerical mesh) to the laboratory
- versus: Lagrangian view where $\Omega = \Omega(t)$ moves

mass conservation (a physics axiom)

$$\varphi(t,x) = \rho(t,x)$$ fluid mass density (scalar)

units: $\frac{\text{mass}}{\text{volume}}$

$$s(t,x) = 0$$ no mass creation/annihilation

$$\vec{u}(t,x)$$ velocity of fluid, (vector)

units: $\frac{\text{distance}}{\text{time}}$

$$\vec{F}(t,x) = \rho(t,x)\,\vec{u}(t,x)$$ mass flux (vector)

units: $\frac{\text{mass}}{\text{area} \cdot \text{time}}$

in other words, physicists assume

$$\frac{d}{dt}\left(\int_\Omega \rho\, dx\right) = -\int_{\partial\Omega} \rho \vec{u}\cdot\hat{n}\, dS + 0$$

for every (fixed) $\Omega \subset \mathbb{R}^3$, as part of any fluid model

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in words: no mass is actually created or lost, so the change in mass in a given volume occurs only by moving mass across the boundary of the volume

• <u>mass conservation</u> in PDE form:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0}$$

a.k.a. "continuity equation"

<u>constant density case</u>:

if $\rho(t,x) = \rho_0 > 0$ is constant then

$$0 + \nabla \cdot (\rho_0 \vec{u}) = 0$$

equivalent

$$\int_{\partial\Omega} \vec{u} \cdot \hat{n} \, dS = 0 \quad \forall \Omega$$

so

$$\boxed{\nabla \cdot \vec{u} = 0}$$

<u>incompressibility</u>

def. suppose $\vec{S} : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ is a vector source function, and

vector-valued general conservation law

$$F : [0,T] \times \mathbb{R}^3 \to (3\times 3 \text{ matrices})$$

is a matrix-valued function.

$\vec{\varphi}(t,x)$ is conserved,

where $\vec{\varphi} : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$, if

$$\boxed{\frac{d}{dt}\left(\int_\Omega \vec{\varphi}\, dx\right) = -\int_{\partial\Omega} F\hat{n}\, ds + \int_\Omega \vec{s}\, dx}$$

Conservation of momentum:

$f = ma$

$\therefore \quad [f] = \dfrac{\text{mass}\cdot\cancel{\text{distance}}}{\text{time}^2\,\text{distance}}$

$$\vec{\varphi}(t,x) = \rho(t,x)\,\vec{u}(t,x) \qquad \text{momentum density}$$

↑ units: $\dfrac{\text{mass}}{\text{area}\cdot\text{time}}$

$$\sigma(t,x) = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \qquad \text{"internal" stress in fluid}$$

$$F(t,x) = \underbrace{(\rho\vec{u})\vec{u}^T}_{\text{sometimes: "}\rho\vec{u}\otimes\vec{u}\text{"}} - \sigma \qquad \text{momentum flux}$$

↑ units: $\dfrac{\text{mass}}{\text{area}\cdot\text{time}} \cdot \dfrac{\text{distance}}{\text{time}}$ [units same as: $\dfrac{\text{force}}{\text{area}}$]

$$\vec{S}(t,x) = \rho\vec{g} \qquad \text{body force (here gravity, an example)}$$

↑ units: $\dfrac{\text{mass}}{\text{volume}} \dfrac{\text{dist}}{\text{time}^2} = \dfrac{\text{mass}}{\text{area}\cdot\text{time}^2}$

Conservation of momentum

$$\frac{d}{dt}\left(\int_{\Omega} \vec{\varphi}\, dx\right) = -\int_{\partial\Omega} F \hat{n}\, ds + \int_{\Omega} \vec{s}\, dx$$

$$\boxed{\frac{d}{dt}\left(\int_{\Omega} \rho \vec{u}\, dx\right) = -\int_{\partial\Omega} \left(\rho \vec{u}\vec{u}^T - \sigma\right) \hat{n}\, ds + \int_{\Omega} \rho \vec{g}\, dx}$$

stresses from inertia, as a momentum flux

stress density within material, as a momentum flux

weight density, as a momentum source

Component-wise

if you work one component at a time then

$\vec{\varphi} = \rho\vec{u}, \quad F = \rho\vec{u}\vec{u}^T - \sigma, \quad \vec{s} = \rho\vec{g}$

this is clearer:

$$\varphi = \rho u_i$$

$$\vec{F} = \rho u_i \vec{u} - \vec{\sigma}_i$$

column i of σ

$$s = \rho g_i$$

thus:

$$\frac{d}{dt}\left(\int_\Omega \rho u_i \, dx\right) = -\int_{\partial\Omega} \rho u_i \vec{u}\cdot\hat{n}\, dS + \int_{\partial\Omega} \vec{\sigma}_i \cdot \hat{n}\, dS + \int_\Omega \rho g_i \, dx$$

component-wise PDE form, from divergence theorem:

$$\frac{\partial}{\partial t}(\rho u_i) + \nabla\cdot(\rho u_i \vec{u}) = \nabla\cdot\vec{\sigma}_i + \rho g_i$$

expand by product rules:

$$\underline{\frac{\partial \rho}{\partial t} u_i} + \rho\frac{\partial u_i}{\partial t} + \nabla u_i\cdot(\rho\vec{u}) + \underline{u_i \nabla\cdot(\rho\vec{u})}$$

$$= \nabla\cdot\vec{\sigma}_i + \rho g_i$$

use <u>mass</u> <u>conservation</u>:

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0$$

$$\rho\left(\frac{\partial u_i}{\partial t} + \vec{u}\cdot\nabla u_i\right) = \nabla\cdot\vec{\sigma}_i + \rho g_i$$

reassemble to full vector/matrix form:

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}\right) = \nabla \cdot \sigma + \rho \vec{g}$$

general model for fluids

ρ = density
$\vec{u}$ = velocity
$\rho\vec{g}$ = body force
σ = internal stresses

mass conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

momentum (and mass) conservation:

$$\rho \left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} \right) = \nabla \cdot \sigma + \rho \vec{g}$$

This is a very common place to start modeling a fluid!

def: given a velocity field $\vec{u}$, for an abstract function $\phi(t,x)$ we call OBSERVATION

$$\boxed{\frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + \vec{u} \cdot \nabla \phi}$$

the <u>material</u> <u>time</u> <u>derivative</u> of ϕ, or the <u>derivative</u> <u>following the fluid</u>

Ex: ① <u>mass</u>

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

$$\Longleftrightarrow \quad \frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho = -\rho \nabla \cdot \vec{u}$$

so mass conservation can be written

$$\frac{d\rho}{dt} = -\rho \nabla \cdot \vec{u}$$

② <u>momentum</u>

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}\right) = \nabla \cdot \sigma + \rho \vec{g}$$

so momentum conservation can be written

$$\rho \frac{d\vec{u}}{dt} = \nabla \cdot \sigma + \rho \vec{g}$$

which everyone associates to

$$m\vec{a} = F$$

(Newton's 2nd law)

$$= F_{viscous} + F_{body}$$

next:

- Navier-Stokes for incompressible and viscous fluids

volunteer possibility?:

conservation of energy, with application

later Bueler lecture:

- "reference configurations," and displacement, strain, velocity
- linear elasticity

model for incompressible, viscous fluid ← i.e. Navier–Stokes

axioms:
① mass (ρ) is conserved

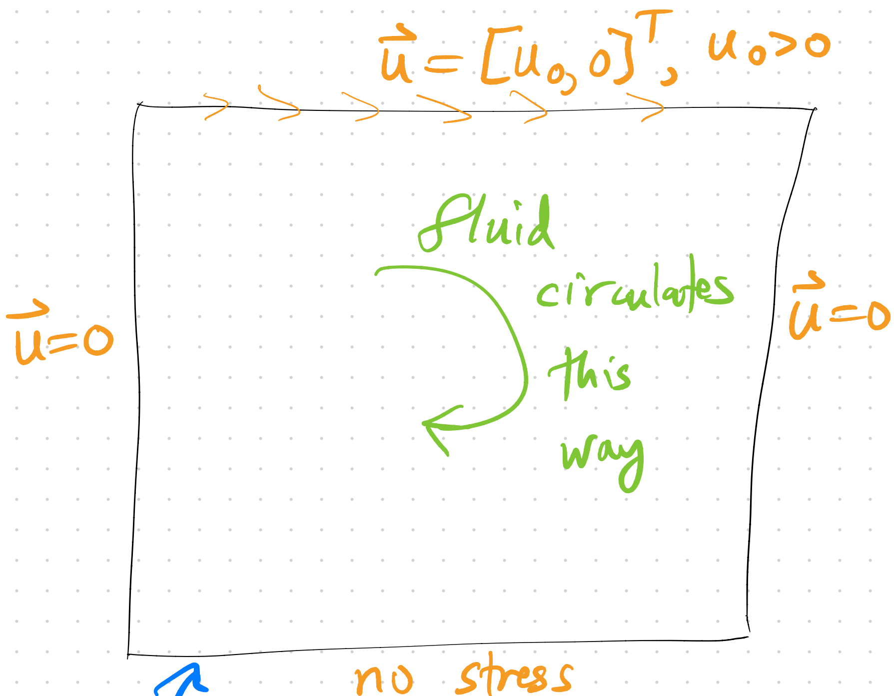
② momentum ($\rho\vec{u}$) is conserved

[③ angular momentum is conserved] this sneaks in ... I will indicate where

implies incompress-ibility { ④ mass density is constant

Newtonian viscous fluid { ⑤ σ has a particular form

demo 2D Navier-Stokes



famous "lid-driven cavity" example

- Firedrake FE solution
- code in bueler.github.io/fluid-solid-seminar/py/bueler/cavity.py
- animated .gif generated from Paraview (via .png)

incompressibility: if $\rho(t,x)=\rho_0>0$ is constant

then

$$\cancel{\frac{\partial \rho}{\partial t}}^{\,0} + \nabla\cdot(\cancel{\rho}_{\rho_0}\vec{u})=0 \quad \leftarrow \text{mass conservation}$$

$$\iff \rho_0 \nabla\cdot\vec{u}=0 \iff \boxed{\nabla\cdot\vec{u}=0}$$

Some people say "$\nabla\cdot\vec{u}=0$" as the axiom (assumption) of incompressibility, but from mass conservation that would seem to allow

$$\frac{d\rho}{dt}=-\rho\nabla\cdot\vec{u}=0$$

so ρ is <u>not</u> constant, but it is (weirdly) preserved as it moves around...

viscous fluid

- to understand viscosity we must consider the stress tensor σ ...
- the main equation we are headed for is

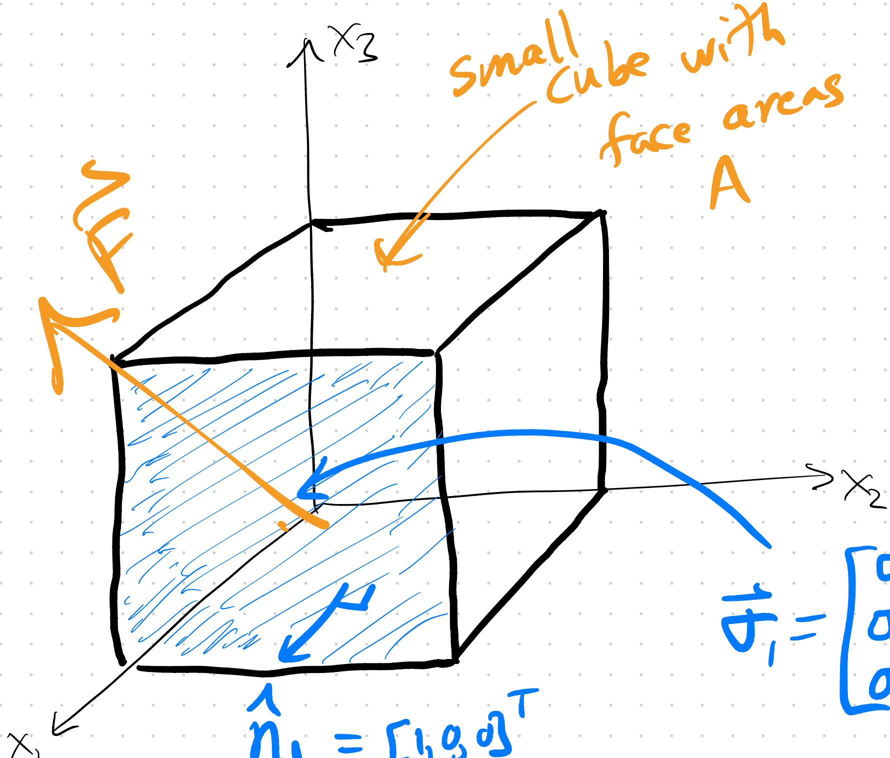
relation between 3x3 matrices

$$\underbrace{\left\{ \sigma = -\overset{\text{pressure}}{p} I + 2\overset{\text{viscosity}}{\nu} \underbrace{D\vec{u}}_{\text{strain rate tensor}} \right.}_{\text{Newton's form (hypothesis) about fluids, in modern notation}}$$

Newton's form (hypothesis) about fluids, in modern notation

picture of $\underbrace{\text{stress}}_{\text{units: } \frac{\text{force}}{\text{area}}}$ tensor:

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \left[\vec{\sigma}_1 \,\middle|\, \vec{\sigma}_2 \,\middle|\, \vec{\sigma}_3 \right]$$



$\hat{n}_1 = [1, 0, 0]^T$

$$\vec{\sigma}_1 = \begin{bmatrix} \sigma_{11} \\ \sigma_{21} \\ \sigma_{31} \end{bmatrix}$$

gives force on this face:

$$\vec{F} = \sigma \hat{n}_1 A = \vec{\sigma}_1 A$$

angular momentum is conserved

$$\sigma = \sigma^T$$ stress tensor is symmetric

derivation in (for example) section 4.3 of E. Tadmor, R. Miller & R. Elliott (2012) <u>Continuum Mechanics and Thermodynamics: From Fundamental Concepts to Governing Equations</u>, Cambridge U. Press

strain rate tensor:

def: given velocity field $\vec{u}$, we have

velocity gradient

$$\nabla \vec{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{bmatrix} = \left[\frac{\partial u_i}{\partial x_j}\right]$$

$$A_s = \frac{1}{2}(A + A^T)$$

strain rate tensor

$$D\vec{u} = \frac{1}{2}(\nabla \vec{u} + \nabla \vec{u}^T) = (\text{symmetric part of } \nabla \vec{u})$$

$$= \begin{bmatrix} \frac{\partial u_1}{\partial x_1} & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) & \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \\ \text{same} & \frac{\partial u_2}{\partial x_2} & \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \\ \text{same} & \text{same} & \frac{\partial u_3}{\partial x_3} \end{bmatrix}$$

- $D\vec{u}$ (and $\nabla\vec{u}$) measure the deformation rates of a small blob of fluid

- for an incompressible fluid, the <u>trace of $D\vec{u}$ is zero</u>:

$$\mathrm{tr}(D\vec{u}) = (D\vec{u})_{11} + (D\vec{u})_{22} + (D\vec{u})_{33}$$

$$= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

$$= \nabla \cdot \vec{u} = 0$$

Newtonian fluid hypothesis (axiom):

there is a scalar field p and another scalar $\nu > 0$ so that

$$\boxed{\sigma \overset{\circledast}{=} -p I + 2\nu D\vec{u}}$$

- p is the <u>pressure</u>
- ν is the (dynamic) viscosity ... usually constant
- $\circledast$ is an axiom about how each small blob of fluid pushes on its neighbors, as it is deformed

- because $\mathrm{tr}(D\vec{u}) = \nabla \cdot \vec{u} = 0$, for an incompressible fluid, ⊗ also gives a formula for the pressure in terms of stress components:

$$0 = \mathrm{tr}(2\nu D\vec{u}) = \mathrm{tr}(\sigma + pI)$$
$$= \mathrm{tr}(\sigma) + p\,\mathrm{tr}(I) = \mathrm{tr}(\sigma) + 3p$$

so

$$\boxed{p = -\frac{1}{3}\mathrm{tr}(\sigma) = -\frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})}$$

which is interpretable in physical terms

the derivation of Navier-Stokes:

- for an incompressible fluid,

$$\nabla \cdot \vec{u} = 0$$ mass conservation

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}\right) = \nabla \cdot \sigma + \rho \vec{g}$$ momentum cons.

- substitute $\sigma = -pI + 2\nu D\vec{u}$ into momentum equation:

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}\right) = -\nabla p + \nabla \cdot (2\nu D\vec{u}) + \rho \vec{g}$$

- <u>optional</u> simplification: if ν constant then $\nabla \cdot (2\nu D\vec{u}) = \nu \nabla^2 \vec{u}$

Navier-Stokes model for an incompressible, linearly-viscous (Newtonian) fluid:

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) - \underbrace{\nabla\cdot(2\nu D\vec{u})}_{\text{if } \nu \text{ constant, } = \nu\nabla^2\vec{u}} + \nabla p = \rho\vec{g}$$

$$\nabla\cdot\vec{u} = 0$$

- this needs boundary and initial conditions!

Optional (but common) form when viscosity is constant

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = \mu \nabla^2 \vec{u} - \frac{1}{\rho} \nabla p + \vec{g}$$

$$\nabla \cdot \vec{u} = 0$$

where $\mu = \frac{\nu}{\rho}$ is kinematic viscosity

- often seen as a nonlinear, constrained, and vector form of the heat equation
- \$1 million prize to show this model is a good one, i.e. mathematically ...