deformation, displacement, strain, and all that

basic kinematic notions for elastic solids

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Fluids & Solids Seminar

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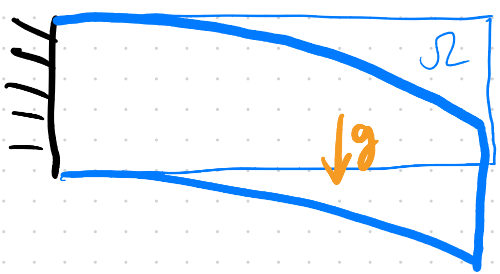
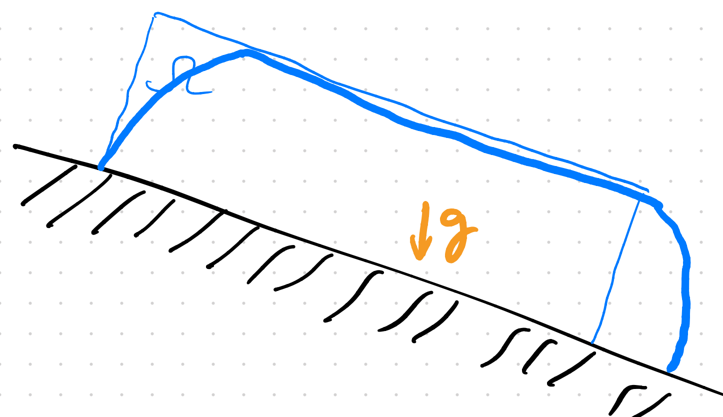
- elastic solids <u>remember</u> where they started
- viscous fluids <u>forget</u> ...
- consider what would happen if you turned off gravity in these situations

<u>elastic beam</u>



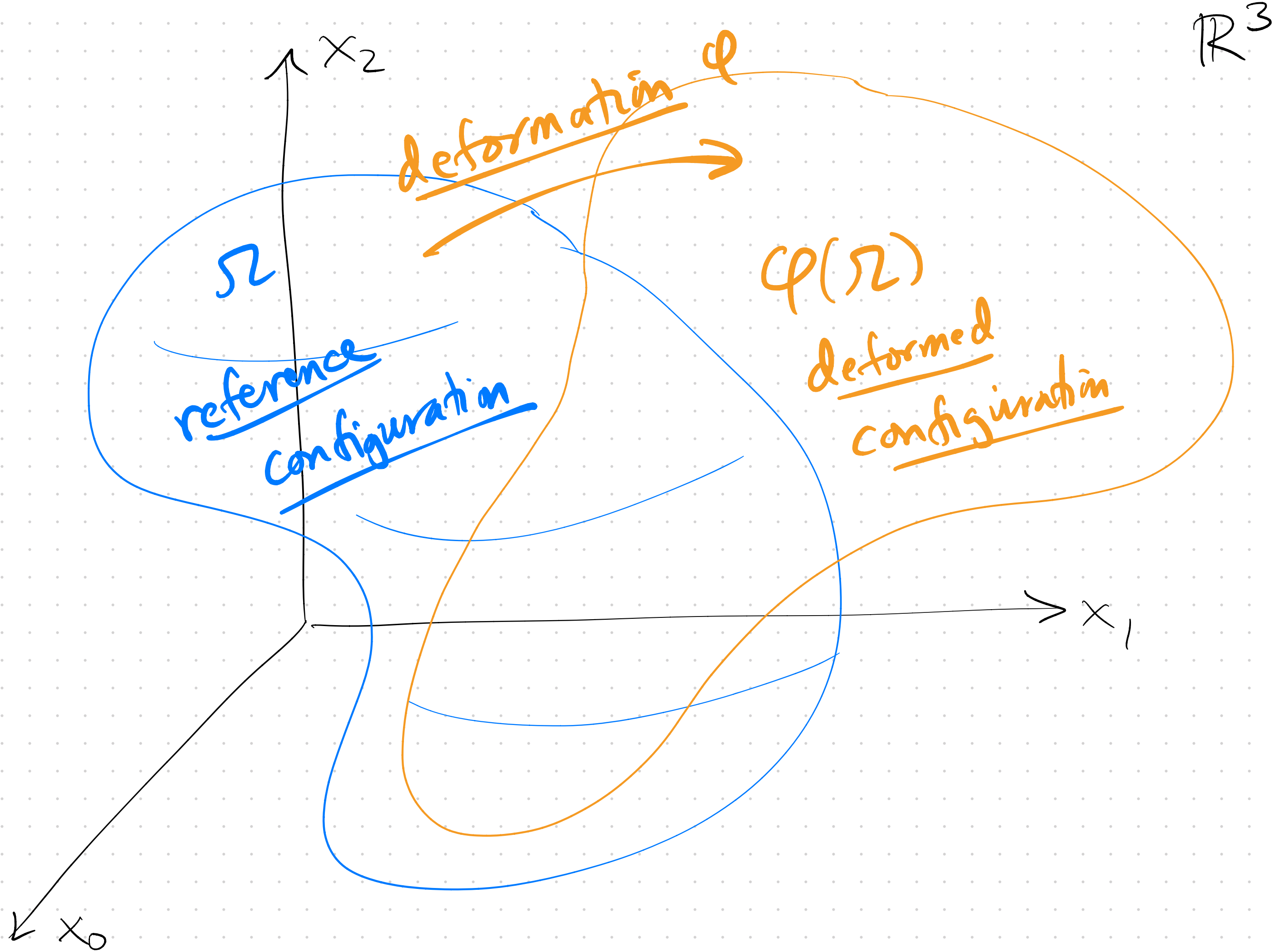
<u>viscous fluid block</u>



Outline:

1. reference configurations, and deformations, in $\mathbb{R}^3$
2. measuring deformation
3. strain
4. stress, constitutive relations, and hyperelasticity

- for elastic solids we are going to compare a reference configuration, a domain in 3D, with its deformed version
- the deformed version might be a (time-independent) equilibrium or static shape, or it could be the changing shape of the elastic solid as it vibrates (time-dependent)



$\mathbb{R}^3$
x_2
x_1
x_0
deformation φ
Ω
reference configuration
$\varphi(\Omega)$
deformed configuration

<u>def</u>: ① a <u>domain</u> in $\mathbb{R}^3$ is an open set $\Omega \subset \mathbb{R}^3$ which is bounded, connected, and has Lipschitz boundary

Ciarlet p. 35

② a <u>deformation</u> of Ω is a continuous, and continuously-differentiable, map $\varphi : \overline{\Omega} \to \mathbb{R}^3$ for which $\det(\nabla\varphi(x)) > 0$ for all $x \in \overline{\Omega}$, and which is injective on Ω

Ciarlet p. 27

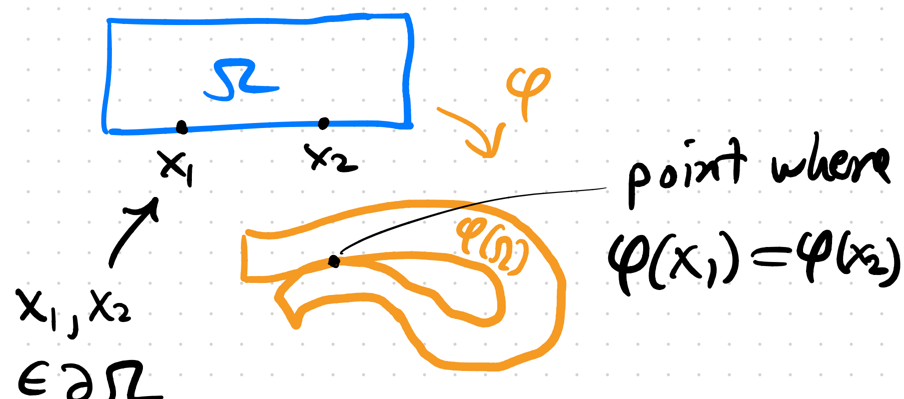
③ we call $\varphi(\Omega)$ the <u>deformed configuration</u>

Why "$\det(\nabla\varphi) > 0$"?

A. if $\det(\nabla\varphi) < 0$ then φ reverses orientation which cannot be done by deformation.
if $\det(\nabla\varphi) = 0$ then deformation is to a point.

Why "injective on Ω" and not "... on $\bar{\Omega}$"?

A. ultimately we do want to allow self-contact



deformation gradient

$$(\nabla\varphi)_{ij} = \frac{\partial \varphi_i(x)}{\partial x_j}$$

or

$$\nabla\varphi = \begin{pmatrix} \frac{\partial\varphi_1}{\partial x_1} & \frac{\partial\varphi_1}{\partial x_2} & \frac{\partial\varphi_1}{\partial x_3} \\ \frac{\partial\varphi_2}{\partial x_1} & \frac{\partial\varphi_2}{\partial x_2} & \frac{\partial\varphi_2}{\partial x_3} \\ \frac{\partial\varphi_3}{\partial x_1} & \frac{\partial\varphi_3}{\partial x_2} & \frac{\partial\varphi_3}{\partial x_3} \end{pmatrix} = \underbrace{F}_{\text{common notation in elasticity texts}}$$

$\nabla\varphi$ is Frechet derivative of φ_j, not its transpose. Ciarlet p. 28

note:

$$d\varphi = F\,dx$$

- often it is easier to describe a deformation not by the final deformed location $\varphi(\Omega)$, but rather by the <u>displacement</u> of the deformation

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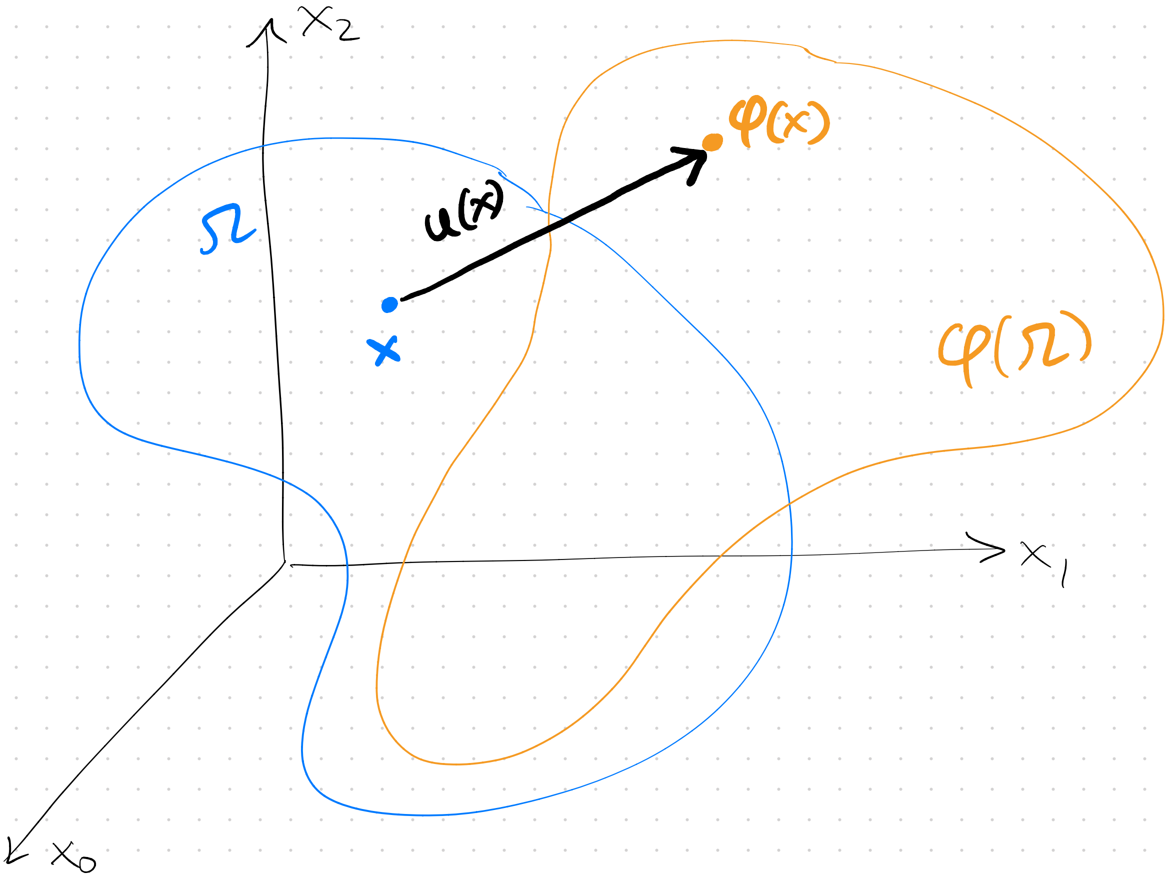
<u>def:</u> for a deformation $\varphi : \bar{\Omega} \to \mathbb{R}^3$, the <u>displacement</u> is $u : \bar{\Omega} \to \mathbb{R}^3$ given by

$$u(x) = \varphi(x) - x$$

also written

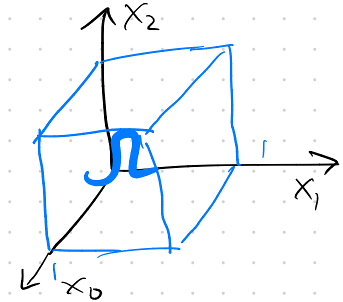
$$\varphi = \mathrm{id} + u$$

$\mathrm{id}(x) = x$ gives $\mathrm{id} : \bar{\Omega} \to \mathbb{R}^3$



x_2
x_1
x_0
Ω
x
$u(x)$
$\varphi(x)$
$\varphi(\Omega)$

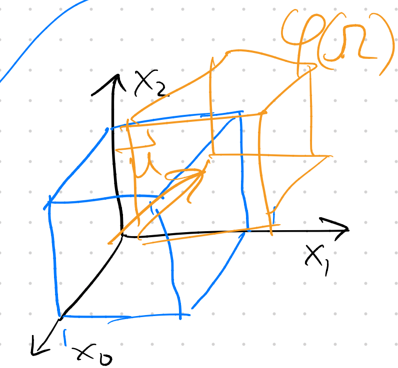
• let's do some examples where



$$\Omega = (\text{unit cube})$$

$$= \{ x = (x_0, x_1, x_2) \in \mathbb{R}^3 : 0 \le x_i \le 1 \}$$

is the reference configuration



__Ex 1__: $\varphi(x) = \left(x_0 - \frac{1}{2},\ x_1 + \frac{1}{2},\ x_2 + \frac{1}{2}\right)$

this is a __translation__ in direction $\underbrace{u}_{\text{constant displacement}} = [-1/2, 1/2, 1/2]^T$

and $\nabla\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$, and $\det(\nabla\varphi) = +1$

Ex 2: $\varphi(x) = (x_0, x_1, -x_2)$

is a <u>reflection</u>, with $u(x) = \varphi(x) - x = [0, 0, -2x_2]^T$,

and $\nabla\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ so $\det(\nabla\varphi) = -1$

... <u>not</u> a deformation!

Ex 3: $\varphi(x) = (x_0, x_2, -x_1)$

is a <u>rotation</u> by 90° about the x_0-axis, with

$u(x) = [0, x_1 - x_2, x_2 + x_1]^T$, and $\nabla\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

so $\det(\nabla\varphi) = +1$

Ex 4: $\varphi(x) = (x_0, x_1, (1-x_1)x_2)$

has $\nabla\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -x_2 & 1-x_1 \end{pmatrix}$ so $\det(\nabla\varphi) = 1-x_1$

so $\det(\nabla\varphi(x)) = 0$ at $x = (\alpha, 1, \gamma) \in \overline{\Omega}$

... **not** a deformation

Ex 5: $\varphi(x) = (x_0 - \frac{1}{2}x_1, x_1, x_2 + \frac{1}{2}x_1)$

has $\nabla\varphi = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}$ so $\det(\nabla\varphi) = +1$;

this is a (volume-preserving) **shear** deformation

Ex 6:

$$\varphi(x) = \Big((1-x_1)x_0 + x_1\left(-0.3\left(x_2-\tfrac{1}{2}\right)+\tfrac{1}{2}\right),\ 2x_1,\ (1-x_1)x_2 + x_1\left(0.3\left(x_0-\tfrac{1}{2}\right)+\tfrac{1}{2}\right) \Big)$$

has

$$\nabla\varphi(x) = \begin{pmatrix} 1-x_1 & -x_0-0.3\left(x_2-\frac{1}{2}\right)+\frac{1}{2} & -0.3x_1 \\ 0 & 2 & 0 \\ 0.3x_1 & -x_2+0.3\left(x_0-\frac{1}{2}\right)+\frac{1}{2} & 1-x_1 \end{pmatrix}$$

... I designed this as a <u>stretch</u> (along x_1 axis) and <u>twist</u> and <u>compress</u>

- so, is your brain not good enough to visualize all this, from these formulas?
- <u>me neither</u>

- use our FE tools, Firedrake & Paraview, just to visualize these Examples 1-6

"but Harry, you have a wand" (Mad-Eye Moody)

- specifically:
  1. mesh the reference configuration, and show as wireframe
  2. use Warp by vector on the displacement $u(x)$
- see code deform.py and Paraview saved state file deform.pvsm

DEMO

formulas to know:

$$\varphi(x) = x + u(x)$$

$$\nabla\varphi(x) = I + \nabla u(x)$$

Q. how does a deformation change volume?

A. $\det(\nabla\varphi(x))$

is the (local) ratio of volumes

$$\det(\nabla\varphi(x)) = \lim_{\delta \to 0} \frac{|\varphi(B_\delta(x))|}{|B_\delta(x)|}$$

$B_\delta(x)$ is a ball around x with radius $\delta > 0$

of a deformed neighborhood of x, over that of the neighborhood itself

Q. how does a deformation change length?

A.
$$\underbrace{\varphi(x+\overset{z\in\mathbb{R}^3}{z})-\varphi(x)}_{\text{vector in } \mathbb{R}^3} = \nabla\varphi(x)\,z + \overset{\text{little o}}{o}(\|z\|)$$

So
$$\|\varphi(x+z)-\varphi(x)\|^2 = (\nabla\varphi(x)z)^T(\nabla\varphi(x)z) + o(\|z\|^2)$$
$$= z^T\left(\nabla\varphi(x)^T\nabla\varphi(x)\right)z + o(\|z\|^2)$$

thus, for small distances ($\|z\|$ small) in Ω, the change in distance is

$$\frac{z^T\left(\nabla\varphi(x)^T\nabla\varphi(x)\right)z}{z^Tz}$$

(recall $\|z\|^2 = z^Tz$)

so

$$C(x) = \nabla\varphi(x)^T \nabla\varphi(x)$$ } "right Cauchy-Green strain tensor" Ciarlet p. 42

is a symmetric, positive-definite matrix which quantifies (local) changes in length

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Ex 5 cont. recall $\varphi(x) = (x_0 - \frac{1}{2}x_1, x_1, x_2 + \frac{1}{2}x_1)$

is a shear deformation with $\nabla\varphi = \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}$ and $\det(\nabla\varphi) = 1$

but: $C(x) = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}$

constant! →

$$= \begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \end{pmatrix} \leftarrow$$

```
>> eig(C)
ans = 0.5
      1.0
      2.0
```

Firedrake Exercise: modify deform.py to color the deformed configuration with the scalar field $\|C(x)\|_2$ for each example

just write $C(x)$ as a TensorFunctionSpace and color by magnitude in Paraview

Theorem (Ciarlet Theorem 1.8-1, p. 44)

if $\varphi : \bar{\Omega} \to \mathbb{R}^3$ is a deformation for which

$$C(x) = \nabla\varphi(x)^T \nabla\varphi(x) = I$$

for all $x \in \Omega$ then φ is a <u>rigid deformation</u>

that is, there exists $a \in \mathbb{R}^3$ and $Q \in \mathbb{R}^{3\times3}$, an orthogonal matrix with $Q^*Q = I$ and $\det(Q) = +1$ (so Q is <u>not</u> a reflection) so that

$$\varphi(x) = a + Qx$$

and $\nabla\varphi(x) = Q$

most important idea of elasticity(?):

} my beginner's opinion...

rigid deformations are <u>not</u> the subject of elasticity theory, which instead assigns an "elastic energy" cost to the <u>strain</u> associated to all the other kinds of deformations

multiple deformations in different contexts

- "separation of concerns" relative to mechanics of rigid bodies

- since $C(x) = \nabla\varphi(x)^T \nabla\varphi(x) = I$ for rigid deformations, we subract-off an I to get the strain relevant to elasticity

<u>def</u>:

$$E(x) = \frac{1}{2}\left(C(x) - I\right)$$ { Ciarlet p. 49

is the <u>strain tensor</u> field, a.k.a. the Green-St. Venant strain tensor

- $E(x)$ is a <u>symmetric</u> matrix, since $C(x)$ is also

- elasticity theory, whether linearized or not, writes this strain tensor in terms of the displacement

coming soon!

Calculation and key formula:

$$2E = C - I = \nabla\varphi^T \nabla\varphi - I$$

recall: $\varphi(x) = x + u(x)$

so $\nabla\varphi = I + \nabla u$

$$= (I+\nabla u)^T (I + \nabla u) - I$$

$$= \cancel{I} + \nabla u + \nabla u^T + \nabla u^T \nabla u \cancel{- I}$$

so

$$\boxed{E(u) = \frac{1}{2}\left(\nabla u(x) + \nabla u(x)^T + \nabla u(x)^T \nabla u(x)\right)}$$

def: the linearized strain tensor is

$$e(u) = \frac{1}{2}\left(\nabla u(x) + \nabla u(x)^T\right),$$

a symmetric matrix

Ex 5 cont. recall $\varphi(x) = \left(x_0 - \frac{1}{2}x_1,\ x_1,\ x_2 + \frac{1}{2}x_1\right)$

is a volume-preserving shear deformation

with $\det(\nabla\varphi(x)) = +1$ (but $C(x) = \nabla\varphi(x)^T \nabla\varphi(x) \neq I$

... not a rigid deformation)

so: $u(x) = \varphi(x) - x = \left(-\frac{1}{2}x_1,\ 0,\ \frac{1}{2}x_1\right)$

from $u(x) = (-\frac{1}{2}x_1, 0, \frac{1}{2}x_1)$

we have

$$\nabla u(x) = \begin{pmatrix} 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

thus

$$e(u) = \frac{1}{2}\left(\nabla u(x) + \nabla u(x)^T\right)$$

$$= \begin{pmatrix} 0 & -\frac{1}{4} & 0 \\ -\frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & 0 \end{pmatrix}$$

- essentially my talk is done; I have presented the <u>kinematics</u> of elasticity, i.e. the part of elasticity theory which is (merely?) describing changes of shape/geometry, and not giving a "why" for those changes
- next are 4 slides on <u>dynamics</u>, where forces (i.e. stresses) appear

stress, and elastic constitutive relations: $e(u(x)) \in \mathbb{R}^{3\times 3}$

- $e(u) = \frac{1}{2}(\nabla u + \nabla u^T) \in \mathbb{R}^{3\times 3}$ is the strain tensor (linearized), coming from spatial derivatives of displacement

Ciarlet p. 286

<u>def</u>: <u>Hooke's law</u> is a constitutive relation which computes the <u>stress tensor</u> $\sigma \in \mathbb{R}^{3\times 3}$ from $e(u)$ and constants $\lambda, \mu > 0$: Lamé parameters

$$\boxed{\sigma = \lambda (\operatorname{tr} e(u)) I + 2\mu\, e(u)}$$

where $\operatorname{tr} M = \sum_{i=1}^{3} M_{ii}$ is the matrix trace

- but we could replace $e(u) \to E(u)$, to make Hooke's law nonlinear, or choose an entirely-different constitutive relation

Q. but what determines stress <u>or</u> strain, noting they are related by a constitutive relation like Hooke's law?

A. boundary (surface) forces, and body forces

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equations of equilibrium:

Ciarlet p. 75

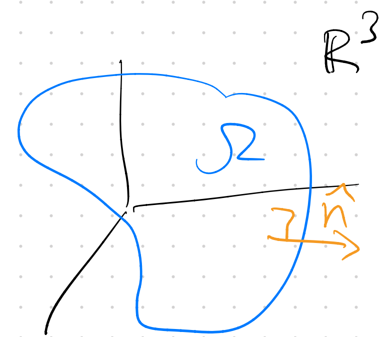
def: the equations of equilibrium for elastic solids are

given body forces

$$-\nabla\cdot\sigma = f \quad \text{in } \Omega$$

given surface forces

$$\sigma\hat{n} = g \quad \text{on } \partial\Omega$$



hyperelasticity:

- many elasticity problems are actually **minimizations**

def: an elastic material is **hyperelastic** if the equations of equilibrium can be written as

Ciarlet Chapter 4

$$\min_{\varphi} I(\varphi) = \int_{\Omega} W(\nabla\varphi)\,dx - \int_{\Omega} f\cdot\varphi\,dx - \int_{\partial\Omega} g\cdot\varphi\,dS$$

for some scalar-valued **energy function** $W : \mathbb{R}^{3\times3} \to \mathbb{R}$

ex: for Hooke's law we can minimize in terms of displacements u (here assuming $g=0$):

see linelas.py

quadratic functional

$$\min_{u} I(u) = \int_{\Omega} \tfrac{1}{2}\lambda\,(\operatorname{tr} e(u))^2 + \mu\, e(u):e(u) - f\cdot u\,dx$$

3 ways elasticity can be nonlinear; ← from Ciarlet's book on finite elements p. 27

1. instead of using the linearized strain tensor $e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$, one returns to the "full" strain tensor $E(u) = \frac{1}{2}(\nabla u + \nabla u^T + \nabla u^T \nabla u)$, which is quadratic in u, in the constitutive relation

2. the constitutive relation could be nonlinear, or (equivalently) the energy function could be non-quadratic

3. instead of minimizing energy over all deformations and/or displacements, we could minimize over a convex subset, as in contact problems

references

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