Fluids & Solids graduate seminar

16 Jan. 2025

(MATH 692, 1.0 credit, crn 35130)

- credit <u>not</u> required!
- if you want 1.0 credit, expect to give one talk

Scope: basic mathematics of continuum models ("continuum mechanics"), for fluid, solids, ...,

and their numerical approximations,

and applications

i.e. anything you want, but with encouragement to *relate it to continuum modeling*

- website at

  bueler.github.io/fluid-solid-seminar

  contains schedule of speakers, links to slides and codes
- my contribution: 2 lectures to start
- volunteers!!? email me with date and title!

yeah Austin!

Continuum mechanics through Navier–Stokes done fast!

plan:

- notation & product rules
- divergence theorem & integration by parts

just calc. 3

- general (integral) conservation ... and its derivatives form

probably new to math grads?

- conservation of mass
- " " momentum

actual physics

next week :(

- Navier–Stokes for incompressible fluids notation:

$x = (x_1, x_2, x_3) \in \mathbb{R}^3$ point

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$ vector ← regarded as column vector

$f: \mathbb{R}^3 \to \mathbb{R}$ scalar function

$\vec{u}: \mathbb{R}^3 \to \mathbb{R}^3$ vector-valued fcn

↖ vector field

$\vec{u} = \begin{bmatrix} u_1(x_1, x_2, x_3) \\ u_2(x_1, x_2, x_3) \\ u_3(x_1, x_2, x_3) \end{bmatrix}$ component functions

casualness assumptions:

1. functions are defined on all of $\mathbb{R}^3$
2. functions are as differentiable as needed

derivatives notation

$$\frac{\partial}{\partial x_i}$$ partial derivative

$$\nabla f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right]^T$$ gradient

$$\nabla \cdot \vec{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$ divergence

dot product

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

product rules: ← audience input ... for scalar & vector functions, dot products, gradient & divergence

$$\nabla(\vec{u}\cdot\vec{v}) = \dots$$

$$\nabla(fg) =$$

$$\nabla\cdot(f\vec{u}) =$$

$$\frac{\partial}{\partial x_i}(fg) =$$

2nd deriv:

$$\nabla\cdot(\nabla f) = \nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2}$$

$$\left.\begin{aligned}
\frac{\partial}{\partial x_i}(fg) &= \frac{\partial f}{\partial x_i} g + f \frac{\partial g}{\partial x_i} \\
\nabla(fg) &= f \nabla g + g \nabla f \\
\nabla \cdot (f\vec{u}) &= \nabla f \cdot \vec{u} + f \nabla \cdot \vec{u}
\end{aligned}\right\} \text{ these we will need today}$$

divergence theorem

also: bounded, connected

$\Omega \subset \mathbb{R}^3$ open set with smooth boundary

$\partial\Omega$ boundary of Ω

$\hat{n}$ outward unit normal vector field along $\partial\Omega$

dx volume element in Ω

dS surface element on $\partial\Omega$

$\vec{u}$ vector field

$\int_a^b f'(x)dx = f(b) - f(a)$

Stokes

$\int_\Omega d\omega = \int_{\partial\Omega} \omega$

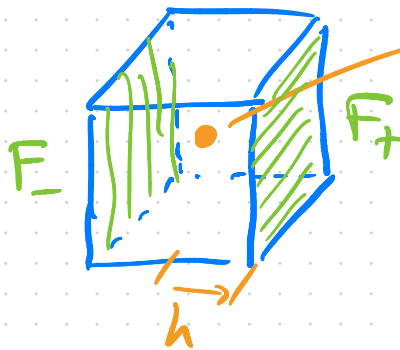
special case

then:

$$\boxed{\int_\Omega \nabla\cdot\vec{u}\,dx = \int_{\partial\Omega} \vec{u}\cdot\hat{n}\,dS}$$

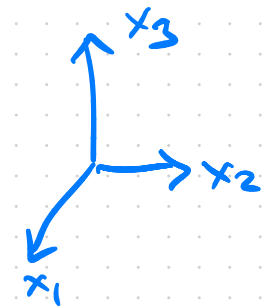
why? true?

$\Omega = C$



center at $y = (y_1, y_2, y_3)$

sides $2h$



note: $$\int_\Omega \nabla \cdot \vec{u}\, dx = \iiint_\Omega \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}\, dx_1\, dx_2\, dx_3$$

consider e.g. $\frac{\partial u_2}{\partial x_2}$ integral:

$$\iiint_C \frac{\partial u_2}{\partial x_2}\, dx_1\, dx_2\, dx_3 = \int_{y_1-h}^{y_1+h} \int_{y_2-h}^{y_2+h} \int_{y_3-h}^{y_3+h} \frac{\partial u_2}{\partial x_2}(x_1, x_2, x_3)\, dx_3\, dx_2\, dx_1$$

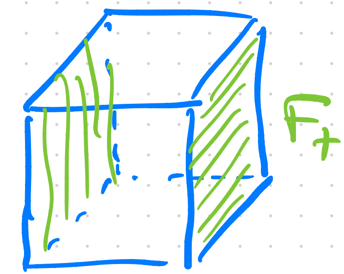
Fubini thm

$$= \int_{y_1-h}^{y_1+h} \int_{y_3-h}^{y_3+h} \left(\int_{y_2-h}^{y_2+h} \frac{\partial u_2}{\partial x_2}(x_1, x_2, x_3)\, dx_2 \right) dx_3\, dx_1,$$

FTC

$$= \int_{..} \int_{..} \Big(u_2(x_1, y_2+h, x_3) - u_2(x_1, y_2-h, x_3) \Big)\, dx_3\, dx_1$$

$$= \int_{F_+} u_2 \underbrace{dx_1\, dx_3}_{= dS \text{ on } F_+} - \int_{F_-} u_2 \underbrace{dx_1\, dx_3}_{= dS \text{ on } F_-}$$



$$= \int_{F_+ \cup F_-} \vec{u} \cdot \hat{n}\, dS$$

where

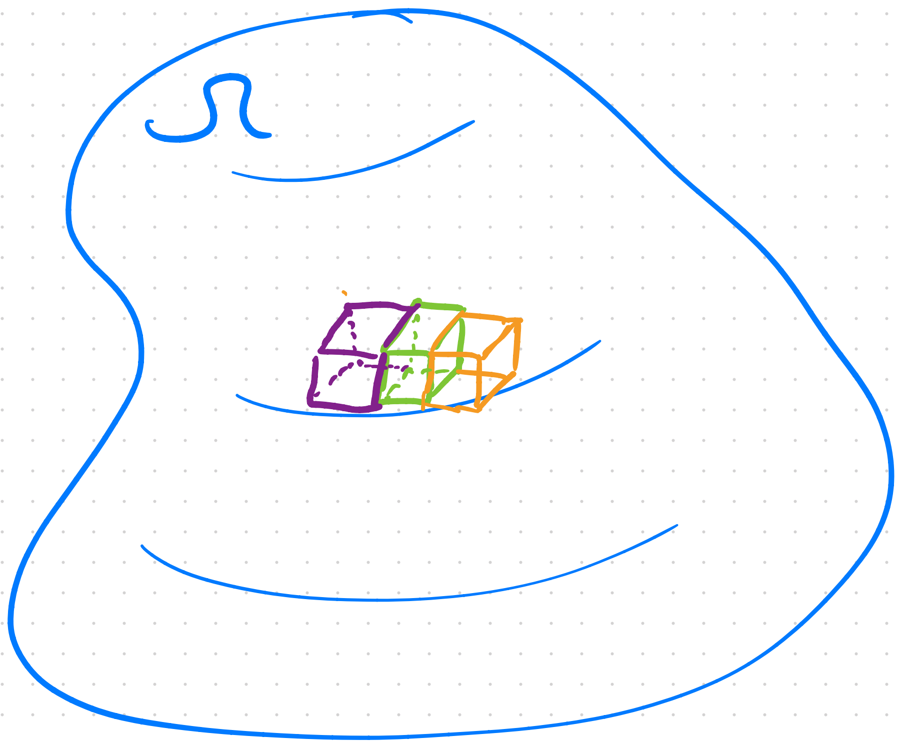
$$\hat{n} = \begin{cases} \langle 0, +1, 0 \rangle & \text{on } F_+ \\ \langle 0, -1, 0 \rangle & \text{on } F_- \end{cases}$$

do same for $\frac{\partial u_1}{\partial x_1}, \frac{\partial u_3}{\partial x_3}$ ... get

$$\int_C \nabla \cdot \vec{u}\, dx = \int_{F_+^1 \cup F_-^1 \cup F_+^2 \cup F_-^2 \cup F_+^3 \cup F_-^3} \vec{u} \cdot \hat{n}\, ds$$

$$= \int_{\partial C} \vec{u} \cdot \hat{n}\, ds$$

extend to general Ω by tiling with cubes, noting cancellation over all interior faces bad picture



integration by parts

← a corollary, sometimes needed, a.k.a. Green's thm

$$\int_{\Omega} \nabla f \cdot \vec{u}\, dx = \int_{\partial\Omega} f \vec{u}\cdot\hat{n}\, ds - \int_{\Omega} f\, \nabla\cdot\vec{u}\, dx$$

proof: ← audience input!

prod. rule

div. thm

$$\nabla\cdot(f\vec{u}) = \nabla f\cdot\vec{u} + f\,\nabla\cdot\vec{u}$$

$$\int_{\partial\Omega} f\vec{u}\cdot\hat{n}\, ds = \int_{\Omega} \nabla\cdot(f\vec{u})\, dx = \int_{\Omega} \nabla f\cdot\vec{u}\, dx + \int_{\Omega} f\,\nabla\cdot\vec{u}\, dx$$

✓

∎

recall

plan:

- notation & product rules
- divergence theorem & integration by parts
- general integral conservation ... and its derivatives form
- conservation of mass
- " " momentum
- Navier–Stokes ...

"general conservation"

<u>def</u>. suppose $s : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$, i.e. $s(t,x)$, is a scalar <u>source</u> function, and $\vec{F} : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$, i.e. $\vec{F}(t,x)$, is a vector-valued <u>flux</u> function. We say

<u>$\varphi(t,x)$ is conserved</u>,

where $\varphi : [0,T] \times \mathbb{R}^3 \to \mathbb{R}$, for given $s, \vec{F}$, if

$$\boxed{\frac{d}{dt}\left(\int_{\Omega} \varphi \, dx\right) = -\int_{\partial\Omega} \vec{F} \cdot \hat{n} \, dS + \int_{\Omega} s \, dx}$$

for all $\Omega \subset \mathbb{R}^3$

φ is conserved means we know how the "amount of φ in Ω", namely

$$\int_\Omega \varphi \, dx$$

changes in time, based on knowing "how much leaves through the boundary" $\left(-\int_{\partial\Omega} \vec{F} \cdot \hat{n} \, dS\right)$ and "how much is created inside" $\left(\int_\Omega s \, dx\right)$

derivative form of (general) conservation

- true over every $\Omega \subset \mathbb{R}^3$:

$$\frac{d}{dt}\left(\int_{\Omega} \varphi \, dx\right) \overset{\circledast}{=} -\int_{\partial\Omega} \vec{F} \cdot \hat{n} \, ds + \int_{\Omega} s \, dx$$

- time derivative:

$$\frac{d}{dt}\left(\int_{\Omega} \varphi \, dx\right) \overset{LDC}{=} \int_{\Omega} \frac{\partial \varphi}{\partial t} \, dx$$

- apply divergence theorem to flux surface integral:

$$\int_{\partial\Omega} \vec{F} \cdot \hat{n} \, ds \overset{DT}{=} \int_{\Omega} \nabla \cdot \vec{F} \, dx$$

- thus rewrite ⊛ as

$$\int_\Omega \left(\frac{\partial \varphi}{\partial t} + \nabla \cdot \vec{F} - S \right) dx = 0$$

- since this is true for every $\Omega \subset \mathbb{R}^3$, we get a partial differential equation (PDE) form of conservation:

$$\boxed{\frac{\partial \varphi}{\partial t} + \nabla \cdot \vec{F} = S}$$

in 641

$$\int_\Omega f \chi_s \, dx = 0 \quad \forall s$$

$$\Rightarrow f = 0 \text{ a.e.}$$

mass conservation (an axiom)

any continuous substance

$\varphi(t,x) = \rho(t,x)$ fluid mass density, scalar

units: $\frac{\text{mass}}{\text{volume}}$

$\rho : [0,T]\times\Omega \to \mathbb{R}$

$s(t,x) = 0$ no mass creation

$\vec{u}(t,x)$ velocity of fluid, vector

units: $\frac{\text{distance}}{\text{time}}$

$\vec{u} : [0,T]\times\Omega \to \mathbb{R}^3$

$\vec{F}(t,x) = \rho(t,x)\,\vec{u}(t,x)$ mass flux

units:

← audience

physicists <u>assume</u>:

$$\frac{d}{dt}\left(\int_{\Omega}\rho\, dx\right) = -\int_{\partial\Omega}\rho\vec{u}\cdot\hat{n}\, dS \text{ "} + 0 \text{ "}$$

$\int_{\Omega} s\, dx = 0$

for every (fixed) $\Omega \subset \mathbb{R}^3$

<u>in words:</u> no mass is actually created or lost, so the change in mass in a given volume occurs only by moving mass across the boundary of the volume

• mass conservation in PDE form:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0}$$

constant density case:

if $\rho(t,x) = \rho_0 > 0$ is constant then

$$0 + \nabla \cdot (\rho_0 \vec{u}) = 0$$

so

$$\boxed{\nabla \cdot \vec{u} = 0}$$

The Eulerian view

- in this (general or mass) conservation view, the "control volume" $\Omega \subset \mathbb{R}^3$ is fixed and does not move
- this view is taken when you fix your coordinates (or numerical mesh) to the laboratory
- versus: Lagrangian view where $\Omega = \Omega(t)$ moves

def. suppose $\vec{S} : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ is a vector source function, and

$$F : [0,T] \times \mathbb{R}^3 \to (3\times 3 \text{ matrices})$$

is a matrix-valued function.

vector-valued!

$\vec{\varphi}(t,x)$ is conserved,

where $\vec{\varphi} : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$, if

$$\boxed{\frac{d}{dt}\left(\int_{\Omega} \vec{\varphi}\, dx\right) = -\int_{\partial\Omega} F \hat{n}\, ds + \int_{\Omega} \vec{s}\, dx}$$

(audience: consistent?)

Conservation of momentum:

$f = ma$

$\therefore \quad [f] = \dfrac{\text{mass} \cdot \text{distance}}{\text{time}^2}$

$$\vec{\varphi}(t,x) = \rho(t,x)\,\vec{u}(t,x)$$ momentum density

units:

$$\sigma(t,x) = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$$ "internal" stress in fluid

$$F(t,x) = \overbrace{(\rho\vec{u})\vec{u}^{T}}^{\text{sometimes: } \rho\vec{u}\otimes\vec{u}} - \sigma$$ momentum flux

units: same as:

$$\vec{S}(t,x) = \rho\,\vec{g}$$ example of source: "body force" from gravity

units:

Conservation of momentum

$$\frac{d}{dt}\left(\int_{\Omega}\vec{\varphi}\,dx\right) = -\int_{\partial\Omega} F\hat{n}\,ds + \int_{\Omega}\vec{s}\,dx$$

$$\boxed{\frac{d}{dt}\left(\int_{\Omega}\rho\vec{u}\,dx\right) = -\int_{\partial\Omega}(\rho\vec{u}\vec{u}^T - \sigma)\hat{n}\,ds + \int_{\Omega}\rho\vec{g}\,dx}$$

$\rho\vec{u}\vec{u}^T$: inertia, as a momentum flux

σ: stress density within material, a momentum flux

$\rho\vec{g}$: weight density, as a momentum source

Component-wise

if you work one component at a time then

$\vec{\varphi} = \rho \vec{u}, \quad F = \rho \vec{u}\vec{u}^T - \sigma, \quad \vec{s} = \rho \vec{g}$

this is clearer:

$$\varphi = \rho u_i$$

$$\vec{F} = \rho u_i \vec{u} - \vec{\sigma}_i$$

column i of σ

$$s = \rho g_i$$

thus:

$$\frac{d}{dt}\left(\int_\Omega \rho u_i \, dx\right) \overset{\circledast}{=} -\int_{\partial\Omega} \rho u_i \vec{u} \cdot \hat{n} \, dS + \int_{\partial\Omega} \vec{\sigma}_i \cdot \hat{n} \, dS$$

$$+ \int_\Omega \rho g_i \, dx$$

component-wise PDE form

$$\frac{\partial}{\partial t}(\rho u_i) + \nabla\cdot(\rho u_i \vec{u}) = \nabla\cdot\vec{\sigma}_i + \rho g_i$$

expand by product rules:

$$\frac{\partial \rho}{\partial t} u_i + \rho \frac{\partial u_i}{\partial t} + \nabla u_i \cdot (\rho\vec{u}) + u_i \nabla\cdot(\rho\vec{u})$$
$$= \nabla\cdot\vec{\sigma}_i + \rho g_i$$

use <u>mass</u> <u>conservation</u>:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0}$$

$$\rho\left(\frac{\partial u_i}{\partial t} + \vec{u}\cdot\nabla u_i\right) = \nabla\cdot\vec{\sigma}_i + \rho g_i$$

back to full vector form:

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}\right) = \nabla \cdot \sigma + \rho \vec{g}$$

general model for fluids

ρ = density
$\vec{u}$ = velocity
$\rho\vec{g}$ = body force
σ = internal stresses

mass conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0$$

momentum (and mass) conservation:

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}\right) = \nabla \cdot \sigma + \rho \vec{g}$$

next:

1. Navier-Stokes for incompressible and viscous fluids

2. "reference configuration," and displacement, strain, velocity

3. linear elasticity