Navier-Stokes solved with finite elements

in Firedrake

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MATH 692 Fluids & Solids Seminar

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assumptions

- all examples here are for 2D domains $\Omega \subset \mathbb{R}^2$
- notation: $\mathbf{u}(t, x, y)$ is velocity and $p(t, x, y)$ is pressure
- density $\rho > 0$ is constant
  - fluid is incompressible
- dynamic viscosity $\mu > 0$ is constant
  - constitutive relation: $\sigma = -pI + 2\mu D\mathbf{u}$
- body force set to zero: $\mathbf{f} = \mathbf{0}$
- units:

$$
\begin{aligned}
[\mathbf{u}] &= \mathrm{m\,s^{-1}} \\
[p] &= \mathrm{N\,m^{-2}} \\
[\rho] &= \mathrm{kg\,m^{-3}} \\
[\mu] &= \mathrm{kg\,m^{-1}\,s^{-1}}
\end{aligned}
$$

the Navier-Stokes model

- the time-dependent, incompressible, constant-viscosity Navier-Stokes equations are:

$$\begin{aligned} \rho\left(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}\right) &= \mu\nabla^2\mathbf{u} - \nabla p && \text{conservation of momentum} \\ \nabla\cdot\mathbf{u} &= 0 && \text{incompressibility (c. of mass)} \end{aligned}$$

potential flow past a cylinder

- recall from 2 weeks ago that Nick provided formula for the potential flow past a cylinder
- in a potential flow the vorticity is zero: $\boldsymbol{\omega} = \nabla \times \mathbf{u} = \mathbf{0}$
- incompressible potential flows satisfy:

$$\nabla \times \mathbf{u} = 0 \qquad \text{conservation of momentum}$$
$$\nabla \cdot \mathbf{u} = 0 \qquad \text{incompressibility (c. of mass)}$$

- zero curl ($\nabla \times \mathbf{u} = 0$) so there exists a potential $\mathbf{u} = \nabla\phi$
- ...and by incompressibility ϕ is harmonic: $\nabla^2\phi = 0$
- assume far field velocity $\mathbf{u} = (U_0, 0)$
- assume non-penetration and free slip on circle $r = a$
- get by separation of variables:

$$\phi(r,\theta) = U_0\left(r + \frac{a^2}{r}\right)\cos(\theta)$$

potential flow past a cylinder

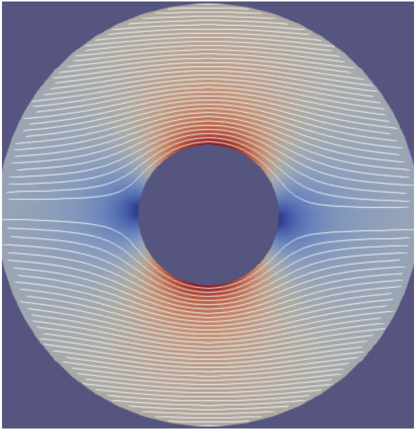
- potential:

$$\phi(r,\theta) = U_0 \left(r + \frac{a^2}{r}\right) \cos\theta$$

- velocity $\mathbf{u} = \nabla\phi$:

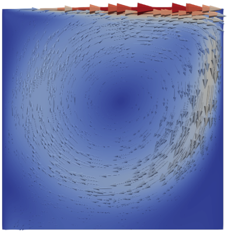
$$\mathbf{u}(r,\theta) = U_0 \left(1 - \frac{a^2}{r^2}\right) \cos\theta\, \hat{\mathbf{r}} - U_0 \left(1 + \frac{a^2}{r^2}\right) \sin\theta\, \hat{\theta}$$

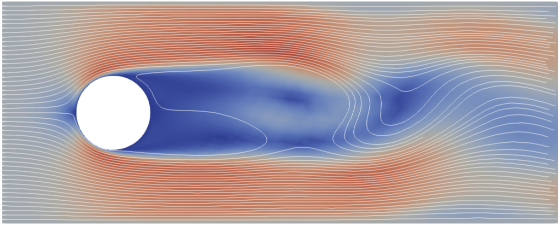
- *however*, the boundary condition along the cylinder, non-penetration and free slip, is not realistic for a viscous fluid
- in fact there is substantial vorticity along the cylinder
- and flow separation . . .



plan for today

- use Firedrake to solve Navier-Stokes (NS) in two situations:
  1. lid-driven cavity on a square ← *DEMO NOW!*
  2. flow around a cylinder on a custom mesh





- TO DO today:
  - the Reynolds scaling argument, to reduce # of parameters
  - implicit discretization of Navier-Stokes (in time)
  - weak form
  - practical Firedrake coding
  - visualization with Paraview
  - meshing with Gmsh

Reynolds number

- recall NS equations:

$$\rho \left(\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} \right) = \mu \nabla^2 \mathbf{u} - \nabla p, \qquad \nabla \cdot \mathbf{u} = 0$$

- a particular simulation sets several scales:

  ρ density
  μ dynamic viscosity
  L (how long and wide is the domain?)
  V (how fast is the fluid, e.g. at boundaries?)

- one can change variables in the NS model using these substitutions:

$$\mathbf{u} = V \tilde{\mathbf{u}}, \qquad p = \rho V^2 \tilde{p}, \qquad \nabla = \frac{1}{L} \tilde{\nabla}, \qquad \frac{\partial}{\partial t} = \frac{V}{L} \frac{\partial}{\partial \tilde{t}}$$

  - the new variables have tildes: $\mathbf{u}, p, x, t \to \tilde{\mathbf{u}}, \tilde{p}, \tilde{x}, \tilde{t}$
  - the new variables are dimensionless

Reynolds number

- one rewrites the NS equations using the tilde variables ...
- drop the tildes:

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = \frac{\mu}{\rho V L}\nabla^2\mathbf{u} - \nabla p, \qquad \nabla\cdot\mathbf{u} = 0$$

- note that the coefficient is dimensionless:

$$\frac{[\mu]}{[\rho][V][L]} = \frac{(\mathrm{kg\,m^{-1}\,s^{-1}})}{(\mathrm{kg\,m^{-3}})(\mathrm{m\,s^{-1}})(\mathrm{m})} = 1$$

Definition

the *Reynold's number* is the dimensionless ratio

$$\mathrm{Re} = \frac{\rho V L}{\mu}$$

Reynolds number

- dimensionless NS equations used from now on ($\text{Re} = \rho VL/\mu > 0$):

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} = \frac{1}{\text{Re}} \nabla^2 \mathbf{u} - \nabla p$$
$$\nabla \cdot \mathbf{u} = 0$$

- if Re is small then viscosity is dominant
- if Re is large then inertia is dominant

implicit time steps

- discretization in time typically uses finite differences
- *key idea*: there is no time derivative in the incompressiblity equation!
  - thus NS equations are really a “differential-algebraic” system in time, and infinitely stiff, thus implicitness is a good idea
- I will use $O(\Delta t)$ backward (implicit) Euler method, highly stable:

$$\mathbf{u}_t \approx \frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t}$$

- suppose $\mathbf{u}^{n-1}$ is known from previous time step, or initial condition
- unknowns are $\mathbf{u}^n$ and p^n:

$$\frac{\mathbf{u}^n - \mathbf{u}^{n-1}}{\Delta t} + \mathbf{u}^n \cdot \nabla \mathbf{u}^n = \frac{1}{\mathrm{Re}} \nabla^2 \mathbf{u}^n - \nabla p^n$$
$$\nabla \cdot \mathbf{u}^n = 0$$

  - these equations are continuous in space

implicit time steps

- cleaner notation $\mathbf{u} = \mathbf{u}^n$, $p = p^n$, $\mathbf{u}^{\text{old}} = \mathbf{u}^{n-1}$
- also clear Δt from denominator ... get:

implicit-step Navier-Stokes equations

$$\mathbf{u} - \mathbf{u}^{\text{old}} + \Delta t \left(\mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{\text{Re}} \nabla^2 \mathbf{u} + \nabla p \right) = 0$$
$$\nabla \cdot \mathbf{u} = 0$$

- we will solve these equations for $\mathbf{u}$, p at every time step, over the domain Ω, using the boundary conditions, and using $\mathbf{u}^{\text{old}}$ from the previous time step or the initial conditions

weak form of NS equations

- above equations are the "strong form", in which $\mathbf{u}$ must have second derivatives and p must have first derivatives
- this is not necessary, and in finite element (FE) method not desirable
- the weak form is built by multiplying by test functions and integrating
- multiply 1st equation by $\mathbf{v}$ and 2nd q, and integrate:

$$\int_\Omega (\mathbf{u} - \mathbf{u}^{\text{old}}) \cdot \mathbf{v}\, dx + \Delta t \int_\Omega \left(\mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{\text{Re}} \nabla^2 \mathbf{u} + \nabla p \right) \cdot \mathbf{v}\, dx = 0$$

$$\int_\Omega (\nabla \cdot \mathbf{u}) q\, dx = 0$$

- integrate by parts to remove 2nd deriv from $\mathbf{u}$ and 1st deriv from p:

$$\begin{aligned}\int_\Omega \left(-\tfrac{1}{\text{Re}} \nabla^2 \mathbf{u} + \nabla p \right) \cdot \mathbf{v}\, dx &= \int_\Omega \nabla \cdot \left[-\tfrac{1}{\text{Re}} (\nabla \mathbf{u}) \mathbf{v} + p \mathbf{v} \right] dx - \int_\Omega -\tfrac{1}{\text{Re}} \nabla \mathbf{u} : \nabla \mathbf{v} + p (\nabla \cdot \mathbf{v})\, dx \\ &= \int_\Omega \tfrac{1}{\text{Re}} \nabla \mathbf{u} : \nabla \mathbf{v} - p (\nabla \cdot \mathbf{v})\, dx + \underbrace{\int_{\partial\Omega} \left(pI - \tfrac{1}{\text{Re}} \nabla \mathbf{u} \right) \hat{\mathbf{n}} \cdot \mathbf{v}\, dS}_{\text{boundary integral}}\end{aligned}$$

boundary conditions

- two boundary conditions considered here, on separate parts of boundary
- stress free (Neumann):

$$\left(pI - \frac{1}{\text{Re}}\nabla \mathbf{u}\right) \hat{\mathbf{n}} = \mathbf{0}$$

  - test functions $\mathbf{v}$ unrestricted on this part of boundary
- given velocity (Dirichlet):

$$\mathbf{v} \text{ given}$$

  - test functions satisfy $\mathbf{v} = \mathbf{0}$ on this part of boundary
- assuming one or the other applies everywhere on boundary, then the boundary integral on last slide is zero

weak form

implicit-step Navier-Stokes equations in weak form

$$\int_\Omega (\mathbf{u} - \mathbf{u}^{\text{old}}) \cdot \mathbf{v}\, dx + \Delta t \int_\Omega \left(\frac{1}{\text{Re}} \nabla \mathbf{u} : \nabla \mathbf{v} + (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{v} - p \nabla \cdot \mathbf{v} \right) dx = 0$$

$$\int_\Omega (\nabla \cdot \mathbf{u}) q\, dx = 0$$

for all test velocities $\mathbf{v}$, with $\mathbf{v} = 0$ on Dirichlet boundary, and for all test pressures q

- the problem is nonlinear in $\mathbf{u}$
- but the weak form is linear in $\mathbf{v}$ and q
- note: no derivatives on p, q, and only first derivatives on $\mathbf{u}, \mathbf{v}$

weak form in Firedrake

$$\int_\Omega (\mathbf{u} - \mathbf{u}^{\text{old}}) \cdot \mathbf{v}\, dx + \Delta t \int_\Omega \left(\frac{1}{\text{Re}} \nabla \mathbf{u} : \nabla \mathbf{v} + (\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{v} - p \nabla \cdot \mathbf{v} \right) dx = 0$$

$$\int_\Omega (\nabla \cdot \mathbf{u}) q \, dx = 0$$

becomes

```
F = dot(u - uold, v) * dx
    + dt * (1.0 / Re) * inner(grad(u), grad(v)) * dx
    + dt * dot(grad(u) * u, v) * dx
    - dt * p * div(v) * dx
    - div(u) * q * dx
```

solver in Firedrake: lid-driven cavity

```
mesh = UnitSquareMesh(32, 32)                  # Firedrake utility mesh

V = VectorFunctionSpace(mesh, "CG", 2)    # P2 x P1 finite elements
W = FunctionSpace(mesh, "CG", 1)
Z = V * W
up = Function(Z)

u, p = split(up)
v, q = TestFunctions(Z)
F = ...                                         # previous slide

x, y = SpatialCoordinate(mesh)
bcs = [DirichletBC(Z.sub(0), as_vector([4 * x * (1-x), 0.0]), (4,)),
       DirichletBC(Z.sub(0), Constant((0.0, 0.0)), (1, 2)),]

t = 0.0
uold.interpolate(as_vector([0.0, 0.0]))  # initial velocity zero
u, p = up.subfunctions
spar = { ... }                                  # for Newton solver
for j in range(N):
    solve(F == 0, up, bcs=bcs, solver_parameters=spar)
    t += dt
    uold.interpolate(u)
```

demonstrations

- Python codes are in `py/bueler/` directory of repository

  `github.com/bueler/fluid-solid-seminar`

- you will need to get Firedrake installed to use these

1. demo: lid-driven cavity on a square
   - look at `navierstokes.py` and `cavity.py`
   - visualize with Paraview
2. demo: flow around a cylinder on a custom mesh
   - build mesh using Gmsh, using geometry-description file `cylinder.geo`
   - look at `navierstokes.py` and `cylinder.py`
   - visualize with Paraview
   - play with Reynold's number