The Cauchy Equilibrium Equations in Linear Elasticity: Derivation and Weak Formulation

Emmanuel A. Azorko

Department of Mathematics and Statistics,
University of Alaska Fairbanks

April 3, 2025

Mathematical Preliminaries and Notation

- **Kronecker Delta**

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- **Divergence Theorem.**

$$\iint_{\partial V} \mathbf{u} \cdot \mathbf{n} \, dS \; = \; \iiint_V (\nabla \cdot \mathbf{u}) \, dx.$$

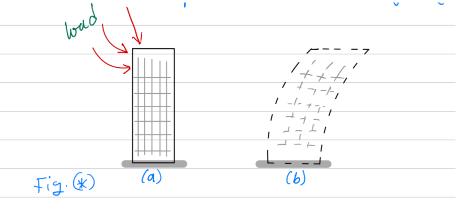
Zero-Value Theorem.

$$\iiint_V f \, dx \; = \; 0 \quad \text{for all } V \quad \Longrightarrow \quad f = 0.$$

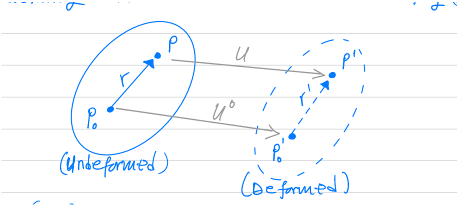
- **Tensor Decomposition**

$$a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji}) = a_{(ij)} + a_{[ij]}$$

Deformation



2-D Deformation example: (a) undeformed (b) Deformed



General deformation of points within a neighborhood

Deformation Map and Displacement Field

We consider the **small deformation theory**.

Let P_0 and P be two neighboring points in the material body, with relative position:

$$\mathbf{r} = \overrightarrow{P_0 P} = P - P_0.$$

Under the deformation map Φ, we have:

$$\Phi(P_0) = P_0', \quad \Phi(P) = P', \quad \text{so that} \quad \mathbf{r}' = \Phi(P) - \Phi(P_0).$$

Define the displacement field:

$$\mathbf{u}(P) = \Phi(P) - P, \quad \text{so} \quad \mathbf{u}^0 = \mathbf{u}(P_0) = \Phi(P_0) - P_0.$$

So $\mathbf{u}(P) = \Phi(P) - P = [\Phi(P) - \Phi(P_0)] + [\Phi(P_0) - P_0] - (P - P_0)$. Then:

$$\mathbf{u}(P) = [\Phi(P) - \Phi(P_0)] + \mathbf{u}^0 - \mathbf{r}.$$

Deformation Map and Displacement Field

Apply a Taylor expansion of Φ about P_0:

$$\Phi(P) - \Phi(P_0) = \nabla\Phi(P_0) \cdot \mathbf{r} + \mathcal{O}(r^2),$$

so the displacement becomes:

$$\mathbf{u}(P) = \nabla\Phi(P_0) \cdot \mathbf{r} + \mathbf{u}^0 - \mathbf{r} + \mathcal{O}(r^2).$$

We define the displacement gradient tensor as:

$$\mathbf{J}(P_0) = \nabla\Phi(P_0) - \mathbf{I},$$

yielding:

$$\mathbf{u}(P) = \mathbf{u}^0 + \mathbf{J}(P_0) \cdot \mathbf{r} + \mathcal{O}(r^2).$$

This expression gives the first-order approximation to the displacement field.

Displacement Gradient Tensor

The Jacobian matrix $\mathbf{J}(\mathbf{P_0})$ is given by:

$$\mathbf{J}(\mathbf{P_0}) = \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} & \frac{\partial u_1}{\partial z} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} & \frac{\partial u_2}{\partial z} \\ \frac{\partial u_3}{\partial x} & \frac{\partial u_3}{\partial y} & \frac{\partial u_3}{\partial z} \end{pmatrix}.$$

Then, the change in relative displacement is

$$\Delta \mathbf{r} = \mathbf{r}' - \mathbf{r} = \mathbf{u} - \mathbf{u}^0 \approx \mathbf{J}(\mathbf{P_0})\,\mathbf{r}.$$

In index notation:

$$\Delta r_i = u_{i,j} r_j, \quad \text{where } u_{i,j} = \frac{\partial u_i}{\partial x_j}.$$

The tensor $u_{i,j}$ is called the **displacement gradient**.

Strain and Rotation Tensors

The displacement gradient $u_{i,j}$ is decomposed into symmetric and antisymmetric parts:

$$u_{i,j} = e_{ij} + \omega_{ij},$$

where

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}).$$

- e_{ij}: **(Linearized) Strain tensor** — measures deformation
- ω_{ij}: **Rotation tensor** — captures rigid-body motion

Since rigid-body motion does not contribute to strain, we focus solely on e_{ij} for stress analysis.

Strain Components

Given an element of a body with dimensions dx, dy, and dz, the normal strain components are:

$$\varepsilon_x = \frac{\partial u_1}{\partial x}, \quad \varepsilon_y = \frac{\partial u_2}{\partial y}, \quad \varepsilon_z = \frac{\partial u_3}{\partial z}$$

The corresponding shear strain components are:

$$\gamma_{xy} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} = \gamma_{yx}$$

$$\gamma_{xz} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \gamma_{zx}$$

$$\gamma_{yz} = \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y} = \gamma_{zy}$$

Compact Notation and Symmetry

The strain tensor

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

in vector notation is:

$$\mathbf{e} = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)$$

Defining $e_x = \varepsilon_x$, $e_y = \varepsilon_y$, $e_z = \varepsilon_z$, and $e_{kl} = \frac{1}{2}\gamma_{kl}$.

It can be shown that e_{ij} is symmetric (Sadd, 2009), and the strain tensor becomes:

$$\mathbf{e} = \begin{pmatrix} e_x & e_{xy} & e_{xz} \\ e_{xy} & e_y & e_{yz} \\ e_{xz} & e_{yz} & e_z \end{pmatrix}$$

Spherical and Deviatoric Strain

The strain tensor e_{ij} is decomposed as:

$$\widetilde{e}_{ij} = \frac{1}{3} e_{kk} \delta_{ij} \qquad \text{[Spherical strain tensor]}$$

$$\hat{e}_{ij} = e_{ij} - \frac{1}{3} e_{kk} \delta_{ij} \qquad \text{[Deviatoric strain tensor]}$$

$$\therefore \quad e_{ij} = \hat{e}_{ij} + \widetilde{e}_{ij}$$

The spherical part corresponds to volumetric deformation, while the deviatoric part reflects shape change.

Newton's Second Law

We consider an elastic body under external load, with mass m and acceleration a. The resultant force at a point in this body must be:

$$\mathbf{F} = m\mathbf{a}.$$

In the static case:

$$\mathbf{F} = \mathbf{0}.$$

Categories/Types of Forces, induced by external load

- Let V be an elastic body.
- **Body Forces**: Proportional to the body's mass. E.g., gravity: $\mathbf{F} = \rho\mathbf{g}$.

$$\mathbf{F}_b = \iiint_V \mathbf{F}(\mathbf{x})\, dx$$

  where $\mathbf{F}(\mathbf{x})$ is the body force density.
- **Surface Forces**: Act on a surface due to physical contact.

$$\mathbf{F}_s = \iint_{\partial V} \mathbf{T}^{(n)}(\mathbf{x})\, dS$$

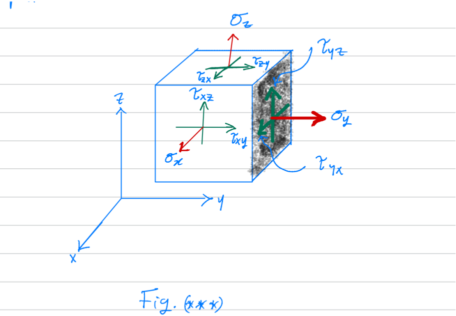
Traction Vector

- $\mathbf{T}^{(n)}(\mathbf{x})$: Traction vector (surface force density).

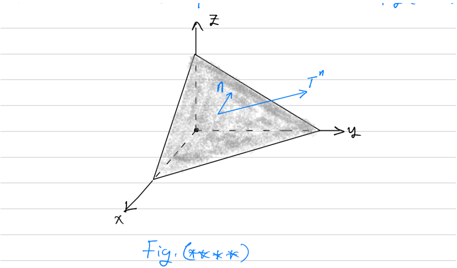
$$\mathbf{T}^{(n)}(\mathbf{x}) = \lim_{\Delta A \to 0} \frac{\Delta \mathbf{F}}{\Delta A}$$

  where $\Delta \mathbf{F}$ is the force applied to a part of the surface of area ΔA.
- Depends on both spatial location and the unit normal vector $\mathbf{n}$.

Stress and Equilibrium



Cubic element



Pyramidal element with arbitrary orientation

Traction in Terms of Components

From figure(***):

$$\mathbf{T}^{(n)}(\mathbf{x}, \mathbf{n} = \mathbf{e}_1) = \sigma_x \mathbf{e}_1 + \tau_{xy} \mathbf{e}_2 + \tau_{xz} \mathbf{e}_3$$
$$\mathbf{T}^{(n)}(\mathbf{x}, \mathbf{n} = \mathbf{e}_2) = \tau_{yx} \mathbf{e}_1 + \sigma_y \mathbf{e}_2 + \tau_{yz} \mathbf{e}_3$$
$$\mathbf{T}^{(n)}(\mathbf{x}, \mathbf{n} = \mathbf{e}_3) = \tau_{zx} \mathbf{e}_1 + \tau_{zy} \mathbf{e}_2 + \sigma_z \mathbf{e}_3$$

Cauchy Stress Tensor

- Normal stresses: σ_x, σ_y, σ_z
- Shear stresses: All others
- Tensor form:

$$\sigma_{ij} = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} \quad \text{[Cauchy stress tensor]}$$

Arbitrary Orientation

In practice, surfaces aren't perfectly aligned.
Consider a pyramidal element with arbitrary orientation:

$$\mathbf{n} = n_1\mathbf{e}_1 + n_2\mathbf{e}_2 + n_3\mathbf{e}_3$$

- n_1, n_2, n_3: Direction cosines of $\mathbf{n}$
- Traction vector:

$$\mathbf{T}^{(n)} = \sum_{k=1}^{3} n_k \mathbf{T}^{(n)}(\mathbf{n} = \mathbf{e}_k)$$

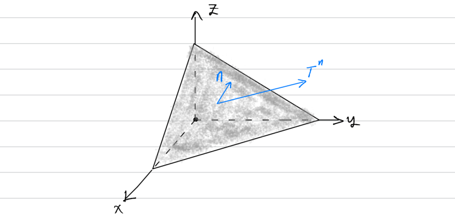


Fig. (****)

Traction in Terms of Stress Tensor

Expanded:

$$\begin{aligned}\mathbf{T}^{(n)} &= (\sigma_x n_1 + \tau_{yx} n_2 + \tau_{zx} n_3)\mathbf{e}_1 \\ &\quad + (\tau_{xy} n_1 + \sigma_y n_2 + \tau_{zy} n_3)\mathbf{e}_2 \\ &\quad + (\tau_{xz} n_1 + \tau_{yz} n_2 + \sigma_z n_3)\mathbf{e}_3\end{aligned}$$

Or compactly:

$$T_i^{(n)} = \sigma_{ji} n_j = \sigma_{ij} n_j$$

Since $\sigma_{ij} = \sigma_{ji}$ (Sadd, 2009).

- This expresses the traction vector $\mathbf{T}^{(n)}$ as a matrix-vector product: the stress tensor acting on the normal vector.
- Justifies using the Cauchy stress tensor to compute surface forces via $\mathbf{T}^{(n)} = \boldsymbol{\sigma} \cdot \mathbf{n}$.

Stress Tensor Decomposition

Like strain, stress can be decomposed:

$$\sigma_{ij} = \tilde{\sigma}_{ij} + \hat{\sigma}_{ij}$$

- **Spherical stress (volumetric)**:

$$\tilde{\sigma}_{ij} = \frac{1}{3}\sigma_{kk}\delta_{ij}$$

- **Deviatoric stress (shape change)**:

$$\hat{\sigma}_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

Principal Stresses and the Spectral Theorem

If the stress tensor $\boldsymbol{\sigma} = (\sigma_{ij})$ is symmetric, then by the **Spectral Theorem**, there exists an orthogonal matrix Q such that:

$$Q^T \boldsymbol{\sigma} Q = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix},$$

where $\sigma_1, \sigma_2, \sigma_3$ are the **eigenvalues** of $\boldsymbol{\sigma}$.

- In this basis, $\boldsymbol{\sigma}$ is diagonal.
- The σ_i are called the **principal stresses**.
- This diagonal form simplifies the analysis of stress.

Cauchy Equilibrium Equations

If a body is in static equilibrium, then:

$$\iint_{\partial V} T_i^n \, ds + \iiint_V F_i \, dx = 0$$
$$\Rightarrow \iint_{\partial V} \sigma_{ij} n_j \, ds + \iiint_V F_i \, dx = 0$$

By divergence theorem:

$$\iiint_V (\sigma_{ij,j} + F_i) \, dv = 0 \Rightarrow \sigma_{ij,j} + F_i = 0$$

This is the **Cauchy equilibrium equation**:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{F} = 0$$

Hooke's Law

- **General (Linear Anisotropic) Case:**

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

where C_{ijkl} is a fourth-order elasticity tensor with **81 entries**. Using symmetry of C_{ijkl}, this reduces to **36 independent constants**(see *Sadd, 2009*).

- **Isotropic Case:**

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

where:

  - λ: Lamé's first parameter
  - μ: Shear modulus (Lamé's second parameter)

- See detailed derivation in *Sadd (2009)*.

Strong Form of the BVP

We consider a fixed reference domain $\Omega \subset \mathbb{R}^n$. The governing equations are:

$$
\begin{aligned}
-\nabla \cdot \boldsymbol{\sigma} = \mathbf{F} \qquad & \text{in } \Omega \\
\boldsymbol{\sigma} = \lambda \, \mathrm{tr}(\varepsilon) \, \delta_{ij} + 2\mu\varepsilon \qquad & \text{in } \Omega \\
\mathbf{u} = \mathbf{g} \qquad & \text{on } \partial\Omega_D \\
\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t} \qquad & \text{on } \partial\Omega_N
\end{aligned}
$$

- $\mathbf{g}$: prescribed displacement on Dirichlet boundary $\partial\Omega_D$
- $\mathbf{t}$: prescribed traction on Neumann boundary $\partial\Omega_N$
- $\mathbf{n}$: outward unit normal on $\partial\Omega$

Weak Form Setup

Multiply the strong form by a test function $\mathbf{v} \in H_0^1(\Omega)$, where $H_0^1(\Omega)$ denotes the Sobolev space of admissible displacements—i.e., vector-valued functions whose components and first derivatives are square-integrable and which vanish on the Dirichlet boundary $\partial\Omega_D$:

$$(-\nabla \cdot \boldsymbol{\sigma}) \cdot \mathbf{v} = \mathbf{f} \cdot \mathbf{v}$$

Integrate over Ω:

$$\int_\Omega (-\nabla \cdot \boldsymbol{\sigma}) \cdot \mathbf{v}\, dx = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx \tag{\#}$$

Component Form

We focus on 2D for clarity; the method extends to 3D similarly.

Let:

$$\boldsymbol{\sigma} = \begin{bmatrix} q_1 & q_2 \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}, \quad \nabla \cdot \boldsymbol{\sigma} = \begin{bmatrix} \nabla \cdot q_1 \\ \nabla \cdot q_2 \end{bmatrix}$$

Then:

$$(\nabla \cdot \boldsymbol{\sigma}) \cdot \mathbf{v} = v_x(\nabla \cdot q_1) + v_y(\nabla \cdot q_2)$$

Substitute into (#):

$$\int_\Omega (-\nabla \cdot \boldsymbol{\sigma}) \cdot \mathbf{v}\, dx = -\int_\Omega [v_x(\nabla \cdot q_1) + v_y(\nabla \cdot q_2)]\, dx \qquad (*)$$

Component Form

Note:

$$\nabla \cdot (v_x q_1) = v_x(\nabla \cdot q_1) + q_1 \cdot \nabla v_x$$
$$\nabla \cdot (v_y q_2) = v_y(\nabla \cdot q_2) + q_2 \cdot \nabla v_y$$

Sub into (*):

$$\begin{aligned}
\int_\Omega (-\nabla \cdot \boldsymbol{\sigma}) \cdot \mathbf{v}\, dx &= -\int_\Omega \left[\nabla \cdot (v_x q_1 + v_y q_2) - q_1 \cdot \nabla v_x - q_2 \cdot \nabla v_y\right] dx \\
&= \int_\Omega (q_1 \cdot \nabla v_x + q_2 \cdot \nabla v_y)\, dx \\
&\quad - \int_\Omega \nabla \cdot (v_x q_1 + v_y q_2)\, dx
\end{aligned}$$

Component Form

By divergence theorem:

$$\int_{\Omega} \nabla \cdot (v_x q_1 + v_y q_2)\, dx = \int_{\partial\Omega_N} (v_x q_1 + v_y q_2) \cdot \mathbf{n}\, ds$$

But:

$$(v_x q_1 + v_y q_2) \cdot \mathbf{n} = (\boldsymbol{\sigma} \cdot \mathbf{v}) \cdot \mathbf{n} = (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \mathbf{v}, \quad \forall \mathbf{v}$$

Hence:

$$\begin{aligned}\int_{\Omega} (-\nabla \cdot \boldsymbol{\sigma}) \cdot \mathbf{v}\, dx = &\int_{\Omega} (q_1 \cdot \nabla v_x + q_2 \cdot \nabla v_y)\, dx \\ &- \int_{\partial\Omega_N} (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \mathbf{v}\, ds\end{aligned}$$

Tensor product notation

Notation: $q_1 \cdot \nabla v_x + q_2 \cdot \nabla v_y = \boldsymbol{\sigma}(\mathbf{u}) : \varepsilon(\mathbf{v})$

Several notations are common for tensor products:
$\mathbf{A} : \mathbf{B} = \mathrm{tr}(\mathbf{A}^T \mathbf{B}) = a_{ij} b_{ij} = \langle \mathbf{A}, \mathbf{B} \rangle$

Final Weak Form

$$\int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \varepsilon(\mathbf{v})\, dx = \int_{\partial\Omega_N} (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \mathbf{v}\, ds + \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dx, \quad \forall \mathbf{v} \in H_0^1(\Omega)$$

- **Goal:** Find $\mathbf{u} \in H^1(\Omega)$ with $\mathbf{u} = \mathbf{g}$ on $\partial\Omega_D$ such that the weak form holds for all $\mathbf{v} \in H_0^1(\Omega)$.
- This is the variational formulation of the elasticity problem with Dirichlet and Neumann boundary data.

Observation

Symmetric and antisymmetric tensors are orthogonal in tensor product, thus:

$$\begin{aligned}\boldsymbol{\sigma} : \nabla \mathbf{v} &= \boldsymbol{\sigma} : \mathrm{sym}(\nabla \mathbf{v}) + \boldsymbol{\sigma} : \text{anti-sym}(\nabla \mathbf{v}) \\ &= \boldsymbol{\sigma} : \varepsilon(\mathbf{v}) + 0\end{aligned}$$

Hence,

$$\boldsymbol{\sigma} : \nabla \mathbf{v} = \boldsymbol{\sigma} : \varepsilon(\mathbf{v})$$

Energy Minimization Formulation

We may define a total potential energy functional:

$$\mathcal{J}(\mathbf{u}) = \int_{\Omega} \left[\frac{1}{2}\lambda(\mathrm{tr}(\varepsilon(\mathbf{u})))^2 + \mu\, \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{u}) - \mathbf{f} \cdot \mathbf{u} \right] dx$$

$$\min_{\mathbf{u}} \mathcal{J}(\mathbf{u}) \iff \int_{\Omega} \boldsymbol{\sigma}(\mathbf{u}) : \varepsilon(\mathbf{v})\, dx = \int_{\partial\Omega_N} (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \mathbf{v}\, ds + \int_{\Omega} \mathbf{f} \cdot \mathbf{v}\, dx \quad \forall \mathbf{v}$$

$$\implies \quad \mu\nabla^2\mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mathbf{f} = 0$$

Weak Formulation in Firedrake

```python
from firedrake import *

# Mesh and function space
mesh = UnitSquareMesh(10, 10)
V = VectorFunctionSpace(mesh, "CG", 1)

# Trial and test functions
u = Function(V, name="Displacement")
v = TestFunction(V)

# Material constants (Lamé parameters)
lambda_ = Constant(1.0)
mu = Constant(1.0)

# Define strain and stress
epsilon = sym(grad(u))
sigma = lambda_ * tr(epsilon) * Identity(2) + 2 * mu * epsilon
```

Weak Formulation in Firedrake

```
# Define source and Neumann term
x, y = SpatialCoordinate(mesh)
f = as_vector([0, exp(-10 * ((x - 0.8)**2 + (y - 0.3)**2))])
t = Constant(as_vector([0.0, 0.0]))

# Weak form: (u) : (v) = f · v + t · v
a = inner(sigma, sym(grad(v))) * dx
L = dot(f, v) * dx + dot(t, v) * ds

# Boundary conditions and solve
bcs = DirichletBC(V, Constant((0.0, 0.0)), "on_boundary")
solve(a == L, u, bcs=bcs)

# Write to file
vtkfile = File("result.pvd")
vtkfile.write(u)
```

Thanks for your attention!!

Questions?

Extra Slide (Proof of notation) I

Notation: $q_1 \cdot \nabla v_x + q_2 \cdot \nabla v_y = \boldsymbol{\sigma}(\mathbf{u}) : \varepsilon(\mathbf{v})$
Proof:

$$
\begin{aligned}
q_1 \cdot \nabla v_x + q_2 \cdot \nabla v_y &= \sigma_x \frac{\partial v_x}{\partial x} + \tau_{xy} \frac{\partial v_x}{\partial y} + \tau_{yx} \frac{\partial v_y}{\partial x} + \sigma_y \frac{\partial v_y}{\partial y} \\
&= \sigma_x \frac{\partial v_x}{\partial x} + 2\tau_{xy} \cdot \frac{1}{2} \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) + \sigma_y \frac{\partial v_y}{\partial y} \\
&= \boldsymbol{\sigma}(\mathbf{u}) : \varepsilon(\mathbf{v})
\end{aligned}
$$

Extra Slide (Notation) I

- **Eigenvalue Problem**

$$a_{ij} n_j = \lambda n_i, \quad (a_{ij} - \lambda \delta_{ij}) n_j = 0$$

- **Characteristic Polynomial**

$$\det(a_{ij} - \lambda \delta_{ij}) = -\lambda^3 + I_a \lambda^2 - II_a \lambda + III_a = 0$$

- **Principal Invariants**

$$I_a = a_{11} + a_{22} + a_{33}, \quad II_a = M_1 + M_2 + M_3, \quad III_a = \det(a_{ij})$$

$$M_1 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad M_2 = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad M_3 = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix}$$

- **Principal Values Classification**

$$\lambda_1 \neq \lambda_2 \neq \lambda_3, \quad \lambda_1 \neq \lambda_2 = \lambda_3, \quad \lambda_1 = \lambda_2 = \lambda_3 \text{ (Isotropic)}$$

Extra Slide (Notation) I

- **Diagonal Form in Principal Axes (Sadd, 2009)**

$$a_{ij} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

- **Invariants in Diagonal Form**

$$\overline{I}_a = \lambda_1 + \lambda_2 + \lambda_3$$

$$\overline{II}_a = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3$$

$$\overline{III}_a = \lambda_1\lambda_2\lambda_3$$

- **Principal Value Ordering**

$$\lambda_1 > \lambda_2 > \lambda_3$$