Conservation and DG

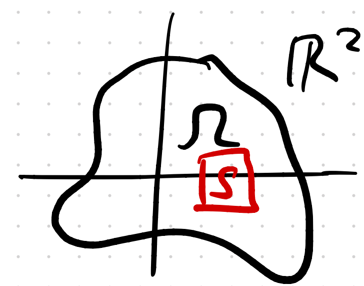
4 April 2024

"discontinuous Galerkin"

- Poisson strong form again:

$$-\nabla^2 u \overset{\circledast}{=} f \quad \text{on } \Omega$$

- recall $\nabla^2 u = \nabla \cdot \nabla u$



integrate $\circledast$ over any $S \subset \Omega$:

$$\int_S -\nabla \cdot \nabla u = \int_S f$$

div. thm

$$-\int_{\partial S} \nabla u \cdot \hat{n} = \int_S f \qquad (1)$$

multiply ⊛ by any v and integrate over Ω:

$$\int_\Omega -(\nabla^2 u)\, v = \int_\Omega f v$$

will assume $u=0$ on Γ_D
and $\nabla u\cdot\hat{n} = g_N$ on Γ_N

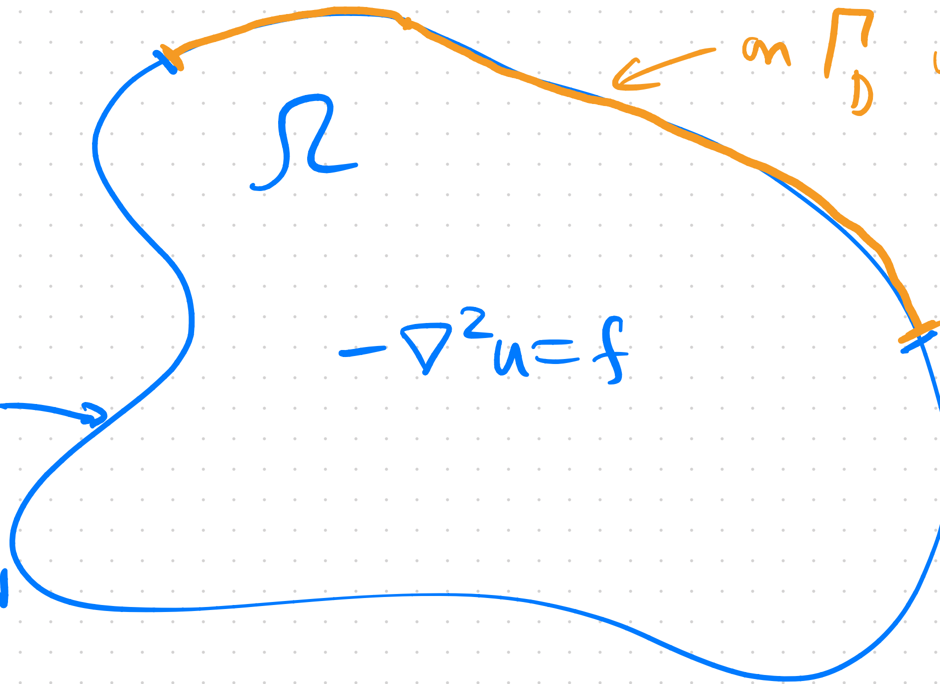
div. thm

$$\int_\Omega \nabla u \cdot \nabla v - \int_{\partial\Omega} v \nabla u \cdot \hat{n} = \int_\Omega f v \qquad ②$$

idea: exact solution satisfies both ① (for all S) and ② (for all v)

Q. Which do we want to use as the basis of a numerical scheme?

a convenient Poisson problem:



on Γ_D we have $u = 0$

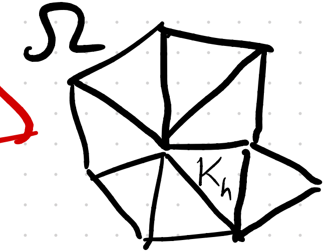
on Γ_N we have $\nabla u \cdot \hat{n} = g_N$

$$\partial\Omega = \Gamma_D \cup \Gamma_N$$

Finite volumes

u = exact soln

u_h = num. soln



$$-\int_{\partial S} \nabla u \cdot \hat{n} = \int_S f \qquad \forall S \subset \Omega$$

approx $\longrightarrow$

$$-\int_{\partial K_h} \nabla u_h \cdot \hat{n} = \int_{K_h} f \qquad \forall \underline{\text{cells}}\ K_h$$

Finite elements

$$\int_\Omega \nabla u \cdot \nabla v - \int_{\Gamma_N} v\, g_N = \int_\Omega f v \qquad \forall v \in H_0^1(\Omega)$$

approx $\longrightarrow$

$$\int_\Omega \nabla u_h \cdot \nabla v_h - \int_{\Gamma_N} v_h\, g_N = \int_\Omega f v_h \qquad \forall v \in \mathcal{V}_h$$

Summary:

strong form

$$-\nabla^2 u = f$$

integrate over K

multiply by v and integrate over Ω

conservation over cells

$$-\int_{\partial K} \nabla u \cdot \hat{n} = \int_K f$$

weak form over Ω

$$\int_\Omega \nabla u \cdot \nabla v - \int_{\Gamma_N} v\, g_N = \int_\Omega f v$$

element-wise weak form:

strong form

$$-\nabla^2 u = f$$

↓ mult. by v and integrate over one element K

element-wise weak form

$$\int_K \nabla u \cdot \nabla v - \int_{\partial K} v \nabla u \cdot \hat{n} = \int_K f v$$

take $v=1$ on K ↙

↓ Sum over elements and cancel along interior edges ("facets")

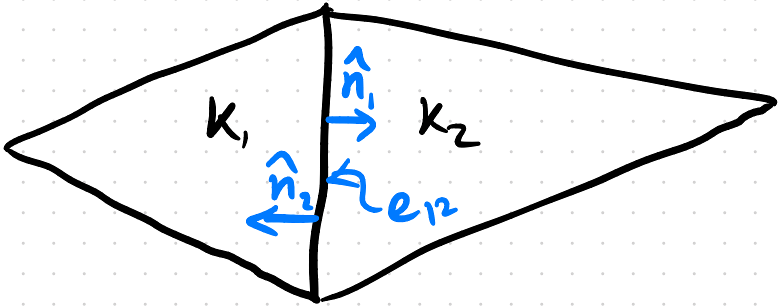
conservation over cells

$$-\int_{\partial K} \nabla u \cdot \hat{n} = \int_K f$$

weak form over Ω

$$\int_\Omega \nabla u \cdot \nabla v - \int_{\Gamma_N} v g_N = \int_\Omega f v$$

re cancelling in adjacent cells: let $S = K_1 \cup K_2$



$$\int_{K_1} \nabla u \cdot \nabla v - \int_{\partial K_1} v \nabla u \cdot \hat{n} = \int_{K_1} f v$$

$$\int_{K_2} \nabla u \cdot \nabla v - \int_{\partial K_2} v \nabla u \cdot \hat{n} = \int_{K_2} f v$$

add and use continuity of ∇u along e_{12}, and $\hat{n}_1 = -\hat{n}_2$:

exact u

$$\int_S \nabla u \cdot \nabla v - \int_{\partial S} v \nabla u \cdot \hat{n}$$

$$= \int_S f v$$

def: a numerical scheme for Poisson equation is conservative if there is an approximation

$$\sigma_h \approx -\nabla u_h$$

for numerical solution u_h, with σ_h single-valued over all edges in $\mathcal{T}_h$, so that

$$\int_{\partial K} \sigma_h \cdot \hat{n} = \int_K f$$

over every element (cell) K in $\mathcal{T}_h$

Lemma: (a) a convergent FE scheme based on the element-wise weak form

$$\int_K \nabla u_h \cdot \nabla v_h + \int_{\partial K} v_h \sigma_h \cdot \hat{n} = \int_K f v_h$$

is conservative if $v_h = \underbrace{\mathbb{1}_K}$ is an allowed test function

characteristic function on K

(b) if an FE scheme is conservative then

(c) $$\int_{\Gamma_D} \sigma_h \cdot \hat{n} = \int_{\Gamma_N} g_N + \int_\Omega f$$ $\Bigg\}$ numerically checkable global conservation demo I will demo 3 Poisson solvers

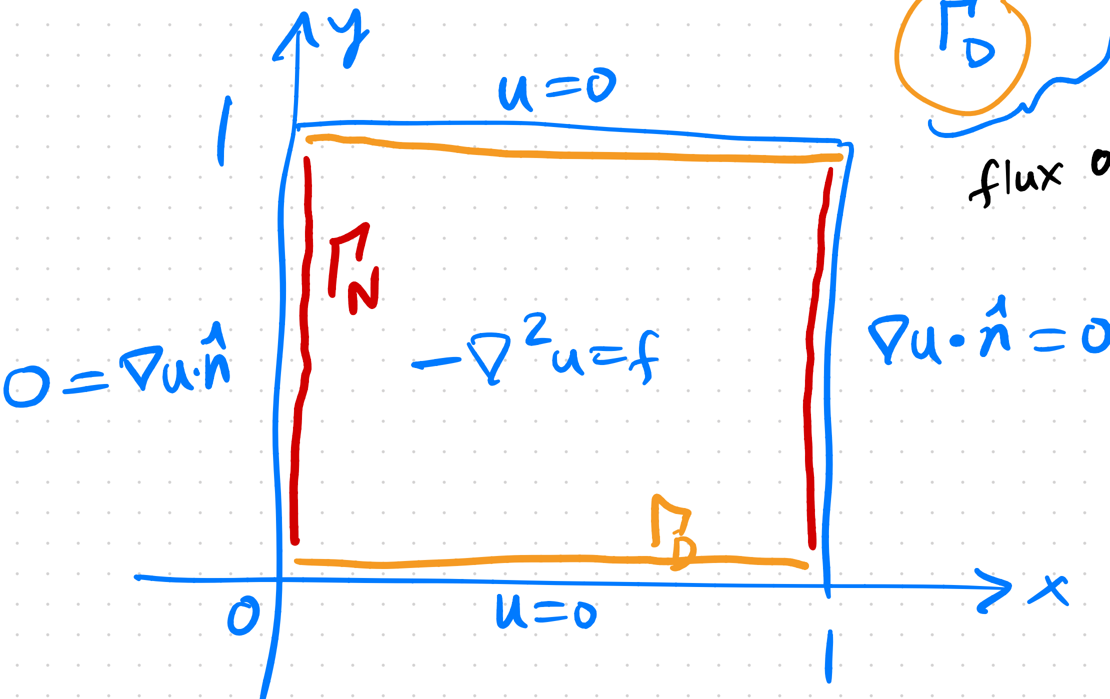
poisson_pCG.py	our usual "primal" CG1 (piecewise-linear) FE scheme, with check on c at end
poisson_pDG.py	"primal" DG0 FE scheme ... and I don't know how to check c
poisson_mDG.py	"mixed" DG0 FE scheme, with check on c at end

- <u>primal</u> means we solve for u_h

- <u>mixed</u> means we solve for u_h <u>and</u> a variable σ_h which approximates $\sigma_h = -\nabla u_h$

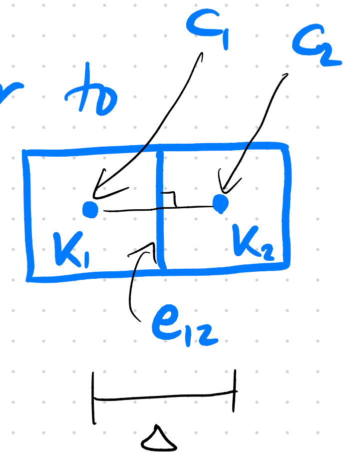
test problem solved by all 3 codes:

$$\underbrace{\int_{\Gamma_D} \sigma_h \cdot \hat{n}}_{\text{flux out}} = \int_{\Gamma_N} \overset{=0 \text{ in this case}}{g_N} + \underbrace{\int_{\Omega} f}_{\text{fint}}$$



primal DG0:

- if faces of cells are perpendicular to lines between cell centers



then

$$\int_{e_{12}\subset\partial K_1} v_h \nabla u_h\cdot\hat{n} + \int_{e_{12}\subset\partial K_2} v_h \nabla u_h\cdot\hat{n}$$

$$\cong \int_{e_{12}} \frac{u_h(c_1)-u_h(c_2)}{\Delta}\left(v_h(c_1)-v_h(c_2)\right)$$

"two-point flux approximation"

- "DGO" means u_h and v_h are constant on each element
- for DGO, $\nabla u_h = 0$ and $\nabla v_h = 0$ in element, so element-wise weak form is just

$$-\int_{\partial K} v_h \nabla u_h \cdot \hat{n} = \int_K f v_h$$

- summing over all elements and using result on previous slide gives this expression:

$$F = (\text{jump}(u) / \text{Delta}) * \text{jump}(v) * \underbrace{dS}_{\text{means integral over interior edges}} - f * v * dx$$

- this is essentially a finite volume method

irritating fact 1: Dirichlet conditions

in this method must be imposed weakly (see code)

irritating fact 2:

I cannot figure out a good way to do a global conservation check:

(c) $\displaystyle\int_{\Gamma_D} \sigma_h \cdot \hat{n} = \int_{\Gamma_N} g_N + \int_{\Omega} f$

not clear how to do this integral

mixed DG:

- quite different, and not as irritating
- again u_h and v_h are constant on each element
  (at least in $k=1$ case below)
- start from strong form $-\nabla^2 u = f$
  but write it as system:

$$\left.\begin{aligned} \sigma &= -\nabla u \\ \nabla\cdot\sigma &= f \end{aligned}\right\} \begin{aligned} &u \text{ scalar} \\ &\sigma = (\sigma_1, \sigma_2) \text{ vector} \end{aligned}$$

- multiply first equation by vector test function ω and second equation by scalar test function v, and integrate over Ω:

$$\int_\Omega \sigma\cdot\omega + \int_\Omega \nabla u\cdot\omega = 0$$

$$\int_\Omega (\nabla\cdot\sigma)\, v = \int_\Omega f v$$

- integrate by parts to move ∇ off of u:

$$\int_\Omega \sigma\cdot\omega - \int_\Omega u(\nabla\cdot\omega) + \int_{\partial\Omega} u\,\omega\cdot\hat{n} = 0$$

$$\int_\Omega (\nabla\cdot\sigma)\, v = \int_\Omega f v$$

- since $u=0$ on Γ_0, that part of boundary integral disappears

Dirichlet becomes natural!

- to enforce $\nabla u \cdot n = -\sigma \cdot n = g_N$ on Γ_N requires a Dirichlet condition on σ:

$$\left.\begin{aligned} \sigma \cdot n &= g_N \\ \omega \cdot n &= 0 \end{aligned}\right\} \text{ on } \Gamma_N$$

Neumann becomes essential!

- <u>final mixed weak form:</u>

$$\left.\begin{aligned} \int_\Omega \sigma \cdot \omega - \int_\Omega u (\nabla \cdot \omega) &= 0 \\ \int_\Omega (\nabla \cdot \sigma) v &= \int_\Omega f v \end{aligned}\right\} \text{ (M)}$$

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F = dot(sigma, omega)*dx - u*div(omega)*dx \
  + div(sigma)*v*dx - f*v*dx
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- solving M accurately/stably requires careful choice of elements:

$$\left.\begin{array}{l} \underbrace{RT_k}_{\sigma,\,\omega \text{ here}} \times \underbrace{DG_{k-1}}_{u,\,v \text{ here}} \\[2em] \overbrace{BDM_k} \times \overbrace{DG_{k-1}} \end{array}\right\} \text{for triangular elements}$$

- see literature and "Periodic Table of the Finite Elements"

results:

- poisson_pCG.py is as expected, but when we measure global conservation ©, it is off by much more than rounding error

try varying m, k

- poisson_pDG.py seems to work, but no apparent way to check ©
- poisson_mDG.py works well and global conservation checks out: © holds to within rounding error

references

"Mixed formulation of Poisson equation"

- firedrakeproject.org/demos/poisson-mixed.py

  explains how poisson_mDG.py works

- github.com/tlroy/thermalporous

  explains how poisson_pDG.py works

see intro/Intro.pdf

Further reading:

- Hughes et al (2000). The continuous Galerkin method is locally conservative. J. Comput. Phys. 163(2), 467-488

- Periodic Table of the Finite Elements, at

  z.umn.edu/femtable