Obstacle Problem FEM

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Overview

- Classical Obstacle Problem
  - Energy Minimization Formulation
  - Variational Inequality Formulation
  - Complementarity Formulation
- Manufacturing a Solution
- Firedrake Implementation

Classical Obstacle Problem

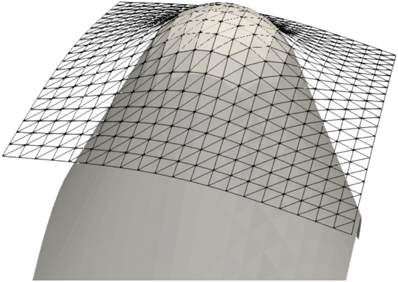


Figure: Discrete Example

Problem: Solve for the displacement of an elastic membrane $u(x, y)$ over a region Ω which minimizes elastic potential energy, subject to a distributed load $f(x, y)$, $u|_{\partial\Omega} g$ and $u \geq \psi$.

Classical Obstacle Problem: Energy Minimization Formulation

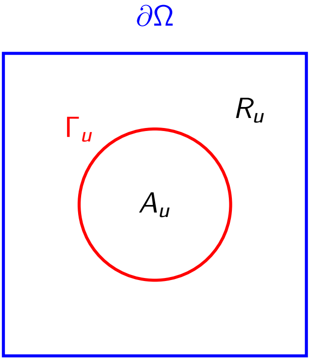
- Elastic Potential Energy Functional:

$$I[v] := \int_{\Omega} \frac{1}{2}|\nabla v|^2 - fv.$$

  - Poisson Equation with Dirichlet Boundary Conditions:
    - ★ $K = \{v \in W^{1,2}(\Omega) : v|_{\partial\Omega} = g\}$
    - ★ Solution: $u = \min_{v \in K} I[v]$
  - Obstacle Problem:
    - ★ $K_{\psi} = \{v \in W^{1,2}(\Omega) : v|_{\partial\Omega} = g, v \geq \psi\}$
    - ★ Solution: $u = \min_{v \in K_{\psi}} I[v]$
- Obstacle Problem is a constrained minimization problem (sometimes).

Classical Obstacle Problem: Energy Minimization Formulation

- The solution u defines the following subsets of Ω
  - Active Set $A_u = \{u = \psi\}$
  - Inactive Set $R_u = \{u > \psi\}$ on which u satisfies a PDE (poisson equation)
  - Free Boundary $\Gamma_u = \partial R_u \cap \Omega$
- $u' = \psi'$ on Γ_u.



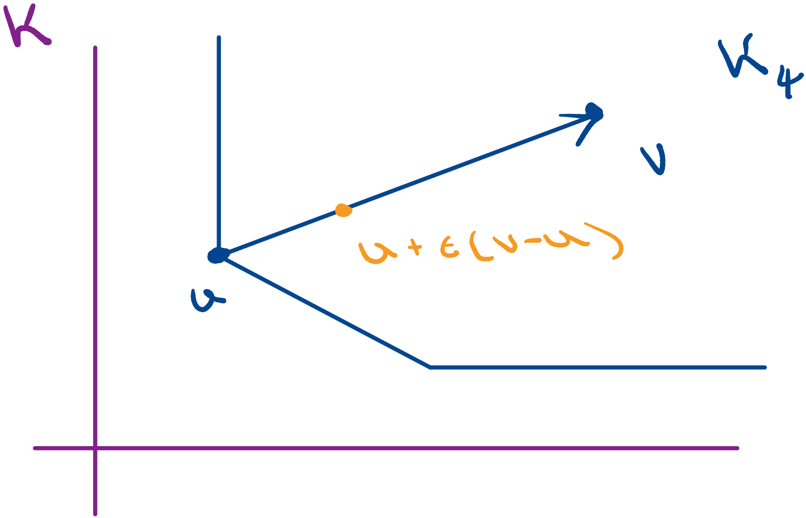
Classical Obstacle Problem: Variational Inequality Formulation

- By linearity of the integral, $I[v]$ is convex over K_ψ and K_ψ is convex.
- Suppose u is the solution to the minimization problem. By convexity of K_ψ, for all $\epsilon \in [0, 1]$ and $v \in K_\psi$ we know that $u + \epsilon(v - u) \in K_\psi$.
- Since u is the minimizer, we know the following directional derivative at u in a *feasible* direction is non-negative.

$$\lim_{\epsilon \to 0^+} \frac{I[u + \epsilon(v - u)] - I[u]}{\epsilon} \geq 0.$$

- Expanding the left hand side gives us a variational inequality (VI).

$$\int_\Omega \nabla u \cdot \nabla(v - u) - \int_\Omega f(v - u) \geq 0, \quad \forall v \in K_\psi.$$


K
K_4
v
u
u + ε(v−u)

Classical Obstacle Problem: Complimentarity Formulation

- The obstacle problem can be written as an complementarity problem (CP).

Find $u \in K$ such that,

$$\begin{cases} -\nabla^2 u - f \geq 0 \\ u - \psi \geq 0, \\ (-\nabla^2 u - f)(u - \psi) = 0. \end{cases}$$

- The last condition is called 'Complementarity'
- Ensures that over all of Ω either the poisson equation (strong form) is being solved or the solution is in contact with the obstacle.

Classical Obstacle Problem: Energy Minimization Formulation

- Energy Minimization Formulation can be helpful for understanding the physics of the problem, and the framing as a constrained minimization problem.
- Not every obstacle problem can be solved by minimizing energy.

$$\int_\Omega F(u, |\nabla u|)\nabla u \cdot \nabla(v - u) \geq \int_\Omega f(v - u), \quad \forall u \in K.$$

  - General form VI, K a convex subset of a Sobelev space.
- Our solver (`vinewtonrsls`) is designed for nonlinear complementarity problems.

Manufacturing a Solution

- Let $\Omega = (-2, 2)^2$, $f = 0$.

- Define the obstacle $\psi(r)$ as the following,

$$\psi(r) = \begin{cases} \sqrt{1 - r^2} & \text{if } r \leq r_0, \\ \ell(r) & \text{if } r > r_0. \end{cases}$$

  Where $r = \sqrt{x^2 + y^2}$, $r_0 = .9$ and $\ell(r) = \psi(r_0) + \psi(r_0)(r - r_0)$.

- ψ is a hemisphere of radius 1 with a linear and continuous differentiable extension from $r = .9$ and onwards.

Manufacturing a Solution

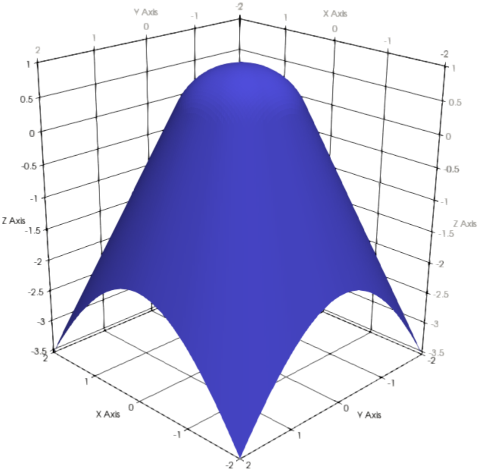


Figure: Proposed ψ

Manufacturing a Solution

- For an inactive set R_u we know that a solution would satisfy the Poisson equation,

$$\nabla^2 u = 0, \quad \text{on } R_u.$$

The ∇^2 operator in polar coordinates is given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

- Since ψ is radially symmetric, the solution is radially symmetric so the PDE simplifies to the following ODE,

$$ru''(r) + u'(r) = 0.$$

Manufacturing a Solution

- Let a be the radial distance from the origin to the free boundary Γ.
- We will enforce $u(2) = 0$ (radial dirichlet boundary conditions)
- Then our manufactured problem becomes solving for u such that,

$$ru''(r) + u'(r) = 0, \quad \text{for } a \leq r \leq 2,$$

with boundary conditions,

$$u(a) = \psi(a), \quad u'(a) = \psi'(a), \quad u(2) = 0.$$

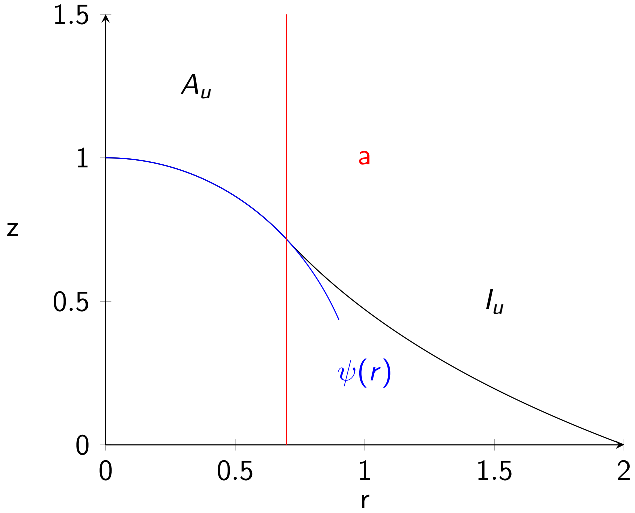


Figure: Radial cross section of $u(r)$.

Manufacturing a Solution

- Such an ODE can be solved analytically and has the form,

$$u(r) = -A\log(r) + B, \quad \text{for } a \leq r \leq 2.$$

- Use $u(2) = 0$ to get B in terms of A, then $u(a) = \psi(a)$ and $u'(a) = \psi'(a)$ becomes a system of equations with two unknowns A and a.
    - $a = 0.697965148223374$
    - $A = 0.680259411891719$
    - $B = 0.471519893402112$
- To get the boundary conditions on Ω we sample $u(r)$ along $\partial\Omega$.

Firedrake Implementation

- This problem is nonlinear so ... snes?
  - `vinewtonrsls` VI-adapted Newton solver with reduces space line search.
  - Solves finite nonlinear complementarity problems with the form,

$$F(w) \geq 0, \quad w \geq 0, \quad F(w)w = 0.$$

- Consider $w = u - \psi$ and $F(w) = -\nabla^2(w - \psi) - f$

$$\begin{aligned} -\nabla^2 u - f &\geq 0 & & F(w) \geq 0 \\ u - \psi &\geq 0 \implies & & w \geq 0. \\ (-\nabla^2 u - f)(u - \psi) &= 0 & & F(w)w = 0 \end{aligned}$$

- In the unconstrained version of this problem, our finite dimensional problem is simply $F(w) = 0$.

Firedrake Implementation

- Current iterate $w^k \in \mathbb{R}^n$.
- Solve for a search direction $d^k \in \mathbb{R}^n$.
  - Identify the nodal (surely) active and (maybe) inactive sets,

$$A(w^k) = \{i \in \{1, ..., N\} | \underbrace{w_i^k = 0 \text{ and } F_i(w^k) > 0}_{\text{surely}}\},$$

$$I(w^k) = \{i \in \{1, ..., N\} | w_i^k > 0 \text{ or } \underbrace{F_i(w^k) \leq 0}_{\text{maybe}}\}.$$

  - Compute search direction on the inactive set,

$$J(w^k)_{I^k, I^k} d_{I_k}^k = -F(w^k)_{I^k}.$$

  - Let,

$$d_i^k = \begin{cases} 0 & \text{if } i \in A(w^k), \\ d_{I_k}^k & \text{if } i \in I(w^k). \end{cases}$$

Firedrake Implementation

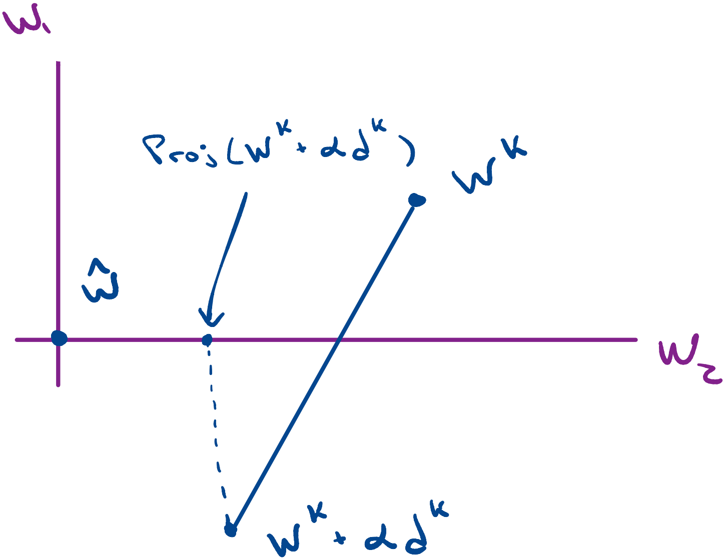
- Line search along $w^k + \alpha d^k$, $\alpha > 0$ is not guaranteed to be admissable ($w \geq 0$).
- Even worse, $F(w)$ (the residual) must be allowed to be positive because of the active set.
- We define the following projection,

$$\text{proj}(w)_i = \begin{cases} 0 & \text{if } w_i^k \leq 0, \\ w_i^k & \text{if } w_i^k > 0. \end{cases}$$

- Then line search is conducted on $\text{proj}(w^k + \alpha d^k)$, which stays admissable ($w \geq 0$)
- We define the following residual,

$$\hat{F}_i(w) = \begin{cases} F_i(w) & \text{if } w_i > 0, \\ \min\{F_i(w), 0\} & \text{if } w_i = 0. \end{cases}$$

- Note $F(w) = 0$, when w is the finite solution.



w1
Proj(wk + αdk)
wk
ŵ
w2
wk + αdk

$$\hat{F}_i(w) = \begin{cases} F_i(w) & \text{if } w_i > 0, \\ \min\{F_i(w), 0\} & \text{if } w_i = 0. \end{cases}$$

If $w_i > 0$, then $\hat{F}_i(w) = F_i(w)$ to solve the Poisson EQ.

If $w_i = 0$ and $F_i(w) \geq 0$, then $\hat{F}_i(w) = 0$, w_i is solved since CP holds.

If $w_i = 0$ and $F_i(w) < 0$, then $\hat{F}_i(w) = F_i(w)$, CP does not hold, w_i is not solved, and we need a nonzero residual.