Waves with Firedrake

Austin Smith
asmith155@alaska.edu
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Outline

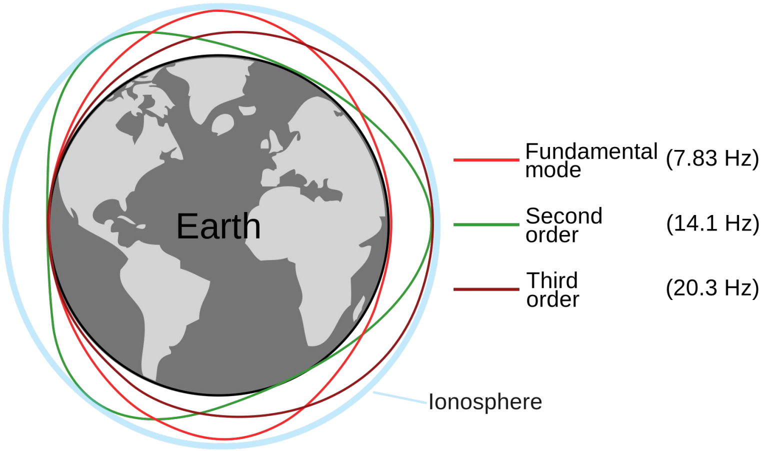
- Motivation
- Background
  - Physical problem
  - Weak form
- Examples
  - Drumhead
  - Resonate box
  - Simple model of Earth-Ionosphere waveguide
- Questions

Motivation

- Time dependent problem
- Not too complicated
    - No vectors
    - No system of equations
    - No magnetodynamics
- Foundation for a Schumann resonance model
    - Applicable at the Earth and beyond
    - FE is a reasonable tool for this question

Background: Schumann Resonances

- Electromagnetic Resonance Frequencies in the Earth-Ionosphere Cavity
  - Frequencies of ~ 7.83, 14.3, 20.8, 27.3, 33.8… Hz
  - Generated by lightning strikes
- First Theorized in 1893 by George FitzGerald [1]
- Theory Modified in 1952 by Winfried Schumann [2]
- Observed in 1960 (M. Balser, and C. A. Wagner) [3]



[1]: Jackson, J. D. (2003), FitzGerald's observation preceded Tesla's, Am. Phys. Soc. News, 17(7), 4.

[2]: Schumann, W. O. (1952), Über die strahlungslosen Eigenschwingungen einer leitenden Kugel, die von einer Luftschicht und einer Ionosphärenhülle umgeben ist (On the free eigen-oscillations of a conducting sphere, surrounded by an air film and an ionospheric shell), Z. Naturforsch. A, 7, 149–154.

[3]: Balser, M., and C. A. Wagner (1960), Observations of Earth-ionosphere cavity resonances, Nature, 188, 638–641.

Image Credit: Vagabard, Wikipedia

Image Date: 2024-04-24

Background: Schumann Resonances

- A useful tool for
  - Analyzing Earth's weather
  - Studying Earth’s electrical environment
- Both created and measured at Earth by
  - Ground receivers
  - Spacecraft [1]
- Theorized to occur on other planets [2]
- A whole project someone could/should do:
  - Generate then measure these resonances via satellites at Earth
    - Verify using known techniques
    - **Develop models**
  - Search for these resonances at other planets to gain insight into their environments

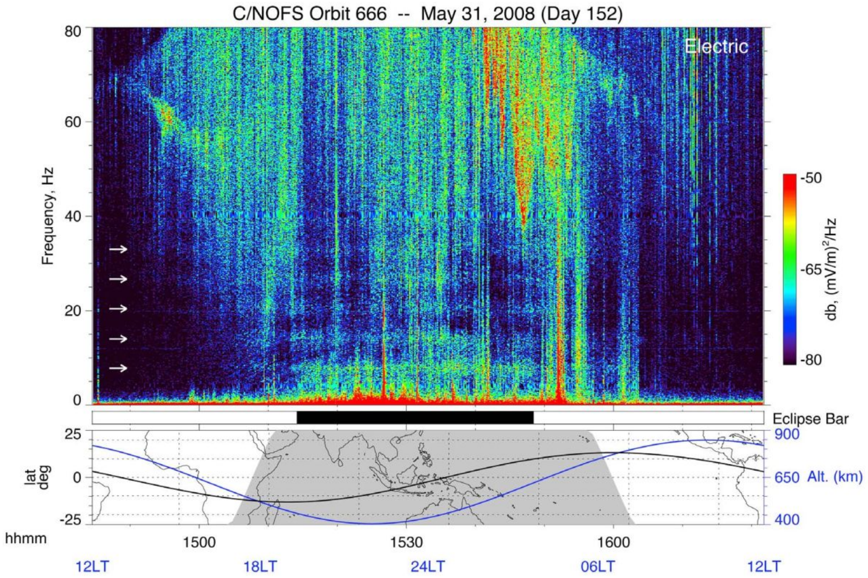


Figure 2 From F. Simões, et al. 2011

[1]: Simões, F., R. Pfaff, and H. Freudenreich (2011), Satellite observations of Schumann resonances in the Earth's ionosphere, Geophys. Res. Lett., 38, L22101, doi:10.1029/2011GL049668.

[2]: Sentman, D. D. 1990, Electrical conductivity of Jupiter's shallow interior and the formation of a resonant planetary ionospheric cavity, Icarus, 88, 73

Background: Equation and Boundary and Initial Conditions

- Wave equation:

$$\frac{\partial^2 \phi}{\partial t^2} - \nabla^2 \phi = 0$$

- For TM modes, electric field is 0 on grounded, conducting surfaces
  - So boundary condition is:

$$\phi = 0 \text{ on } \Gamma_D$$

- Initial conditions vary
  - All take the form of an initial step-function pulse in the domain
  - Excite all wave modes
  - Rely on numerical diffusion and dissipation to only keep resonate modes

Background: Weak Form

- To get to the weak form, first make the substitution:

$$\frac{\partial \phi}{\partial t} = -p$$

- This results in modified equation with a first derivative in time:

$$\frac{\partial p}{\partial t} + \nabla^2 \phi = 0$$

- The weak form of this PDE is:

$$\int_\Omega \frac{\partial p}{\partial t} v \mathrm{d}x = \int_\Omega \nabla \phi \cdot \nabla v \mathrm{d}x$$

Background: Time Stepping

- Verlet method [1]
  - Second-order accurate in time
  - Time reversible
- First step, solve (1) for $\phi^{n+½}$
  - Pointwise operation
- Second step, solve (2) for p^{n+1}
  - Done by inverting mass matrix
  - Linear solver
- Third step, solve (3) for ϕ^{n+1}
  - Pointwise operation

$$\frac{\phi^{n+1/2} - \phi^{n}}{\Delta t/2} = -p^{n} \quad (1)$$

$$\frac{p^{n+1} - p^{n}}{\Delta t} = -\nabla^2 \phi^{n+1/2} \quad (2)$$

$$\frac{\phi^{n+1} - \phi^{n+1/2}}{\Delta t/2} = -p^{n+1} \quad (3)$$

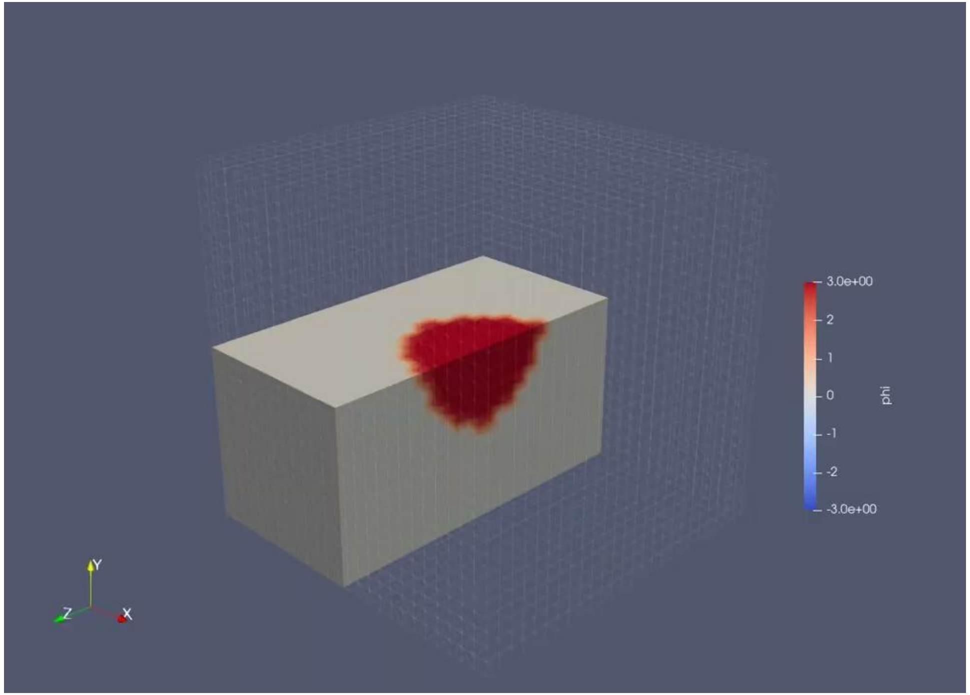
[1]: Benedict J. Leimkuhler, Robert D. Skeel, Symplectic Numerical Integrators in Constrained Hamiltonian Systems, Journal of Computational Physics, Volume 112, Issue 1, 1994, Pages 117-125, ISSN 0021-9991, https://doi.org/10.1006/jcph.1994.1085.

Example 1: Drumhead

- Switch to code and paraview

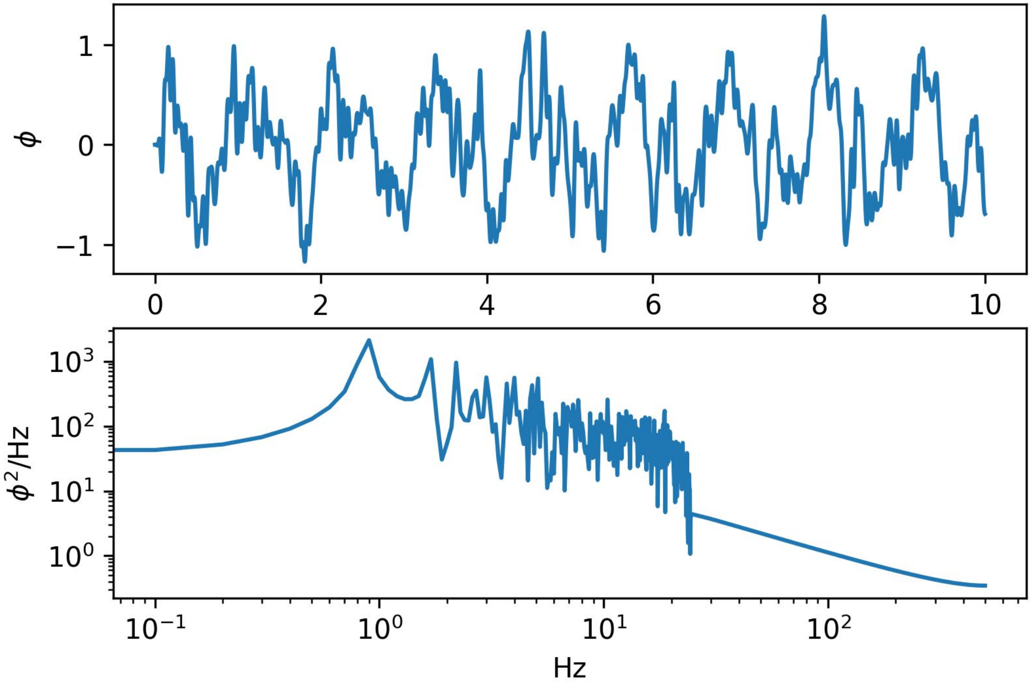
Example 2: Resonate Box

- Look at code
- Took ~15 minutes to run



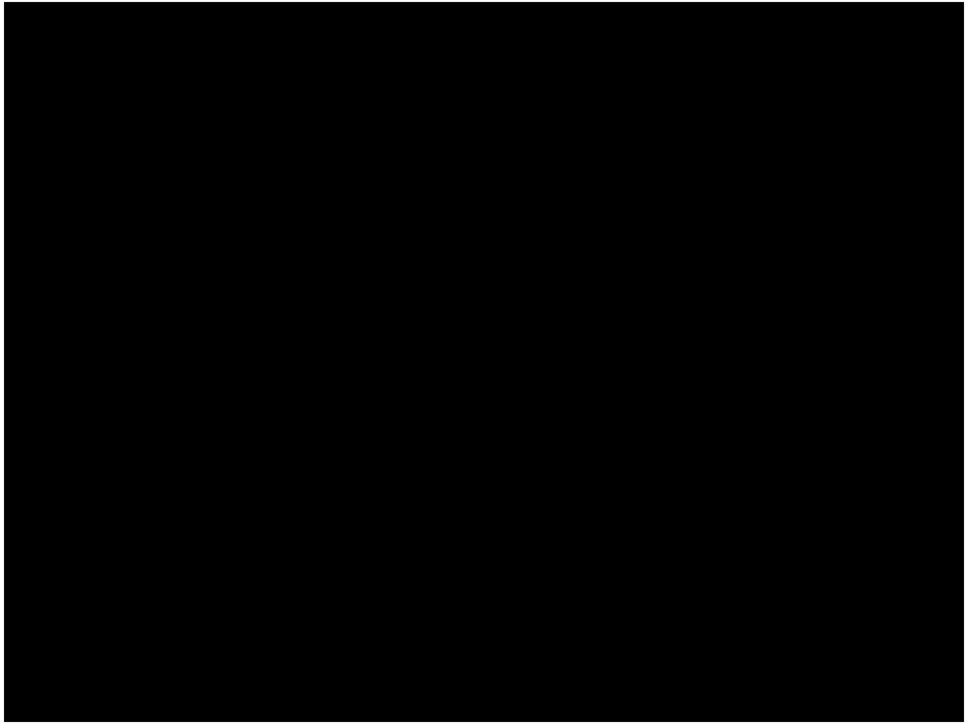
Example 2: Resonate Box

- Extracted a time series from a random point in the domain
  - Took an FFT to see if there is any resonance
  - Maybe at ~1 Hz?



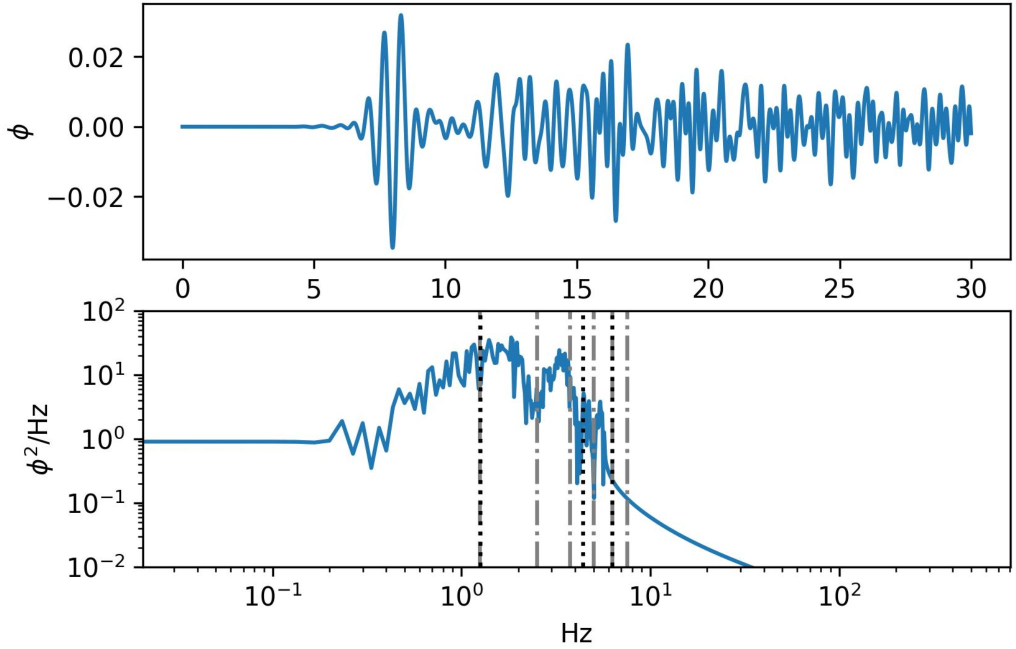
Example 3: Simple Model of Earth-Ionosphere Waveguide

- Look at code
- Took ~6 hours to run



Example 3: Simple Model of Earth-Ionosphere Waveguide

- Extracted a time series from a random point in the domain
    - Took an FFT to see if there is any resonance
    - Black vertical lines show where schumann-like resonances are
    - Grey vertical lines show approximate “parallel plate” resonances



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  - Previous work together and research on Schumann Resonances
- Firedrake tutorial
  - Served as a basis for the three examples

Questions?