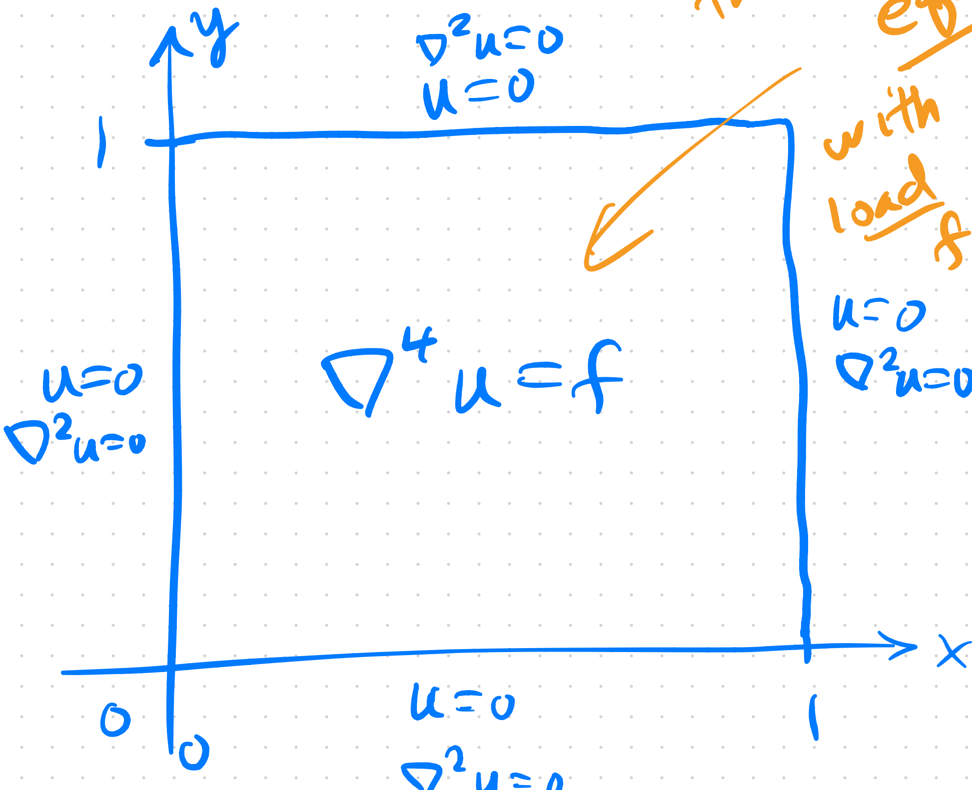
# a system of PDEs

21 March 2024

the plate equation with load $f$



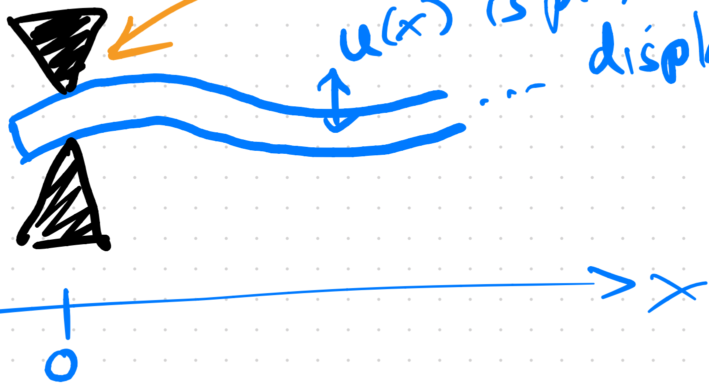

**def:**

$$\nabla^4 u = u_{xxxx} + 2u_{xxyy} + u_{yyyy}$$

is called the **biharmonic operator**

(it is $(\text{Laplacian})^2$)

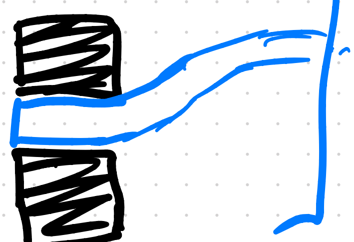
## re. boundary conditions




$u(x)$ is plate displacement ...

"$u=0$ and $\nabla^2 u=0$" corresponds to <u>pinning</u>

[versus <u>clamping</u>


]

... is h

# Firedrake method 1

$$\int_\Omega (\nabla^4 u) v = \dots = \int_\Omega (\nabla^2 u) : (\nabla^2 v) \quad \dots$$

not doing that, although it is possible

# Firedrake method 2

$$\nabla^4 u = -\nabla^2 v \quad \text{where} \quad v = -\nabla^2 u$$

minus so that in sorted we have positive operators

so:

$$\underbrace{\nabla^4 u = f}_{\text{scalar, 4th order}} \iff \underbrace{\begin{aligned} -\nabla^2 v &= f \\ -v - \nabla^2 u &= 0 \end{aligned}}_{\text{system, 2nd order}}$$

we can use CG1 elements

**strong form:**

$$-\nabla^2 v = f$$
$$-v - \nabla^2 u = 0$$

(b.c. $u=0, v=0$ on $\partial\Omega$)

**weak form:** multiply first equation by $r \in H_0^1(\Omega)$ and second by $s \in H_0^1(\Omega)$, and add:

$$-\int_\Omega (\nabla^2 v) r - \int_\Omega f r - \int_\Omega v s - \int_\Omega (\nabla^2 u) s = 0$$

integrate by parts and use $r=0$ & $s=0$ along $\partial\Omega$:

$$\int_\Omega \nabla v \cdot \nabla r - \int_\Omega f r - \int_\Omega v s + \int_\Omega \nabla u \cdot \nabla s = 0$$

so:

$$F = (\,\mathrm{dot}(\mathrm{grad}(v), \mathrm{grad}(r)) - f*r$$
$$- v*s + \mathrm{dot}(\mathrm{grad}(u), \mathrm{grad}(s))\,)*dx$$

now Firedrake builds a linear system...

block structure:

$$\begin{bmatrix} A & 0 \\ -I & A \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} f \\ 0 \end{bmatrix}$$

where

$A \approx -\nabla^2$ is discretized Laplacian

## solver choices: -pc_type fieldsplit

one can precondition by inverting <u>blockwise</u>

$$\begin{bmatrix} A^{-1} & 0 \\ 0 & A^{-1} \end{bmatrix} \begin{bmatrix} A & 0 \\ -I & A \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} A^{-1} & 0 \\ 0 & A^{-1} \end{bmatrix} \begin{bmatrix} f \\ 0 \end{bmatrix}$$

for additive fieldsplit

$$\begin{bmatrix} I & 0 \\ -A^{-1} & I \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} A^{-1}f \\ 0 \end{bmatrix}$$

this linear system is fast to solve

where really "$A^{-1}$" = (apply good preconditioner to that block)

for example, <u>multigrid</u>

# Conclusion: a good option combination

not the only good option combination

snes_type: ksponly
ksp_type: gmres
pc_type: fieldsplit
pc_fieldsplit_type: additive
fieldsplit_0_ksp_type: preonly
fieldsplit_0_pc_type: gamg
fieldsplit_1_ksp_type: preonly
fieldsplit_1_pc_type: gamg

demo!
plate.py