

a nonlinear version of Poisson

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Liouville - Bratu equation

A diagram showing a square domain in the xy -plane. The origin is labeled 0 . The top and right boundaries are labeled $u=0$. The bottom boundary is labeled $u=0$. The left boundary is labeled $u=0$. Inside the square, the equation $\nabla^2 u + \lambda e^u = 0$ is written. An orange arrow points from the text on the right towards the equation.

models steady-state
heat conduction plus
an exponentially growing
exothermic reaction
... i.e. a warm
block of explosive

Strong

$$\boxed{-\nabla^2 u - \lambda e^u = 0}$$

← seek $u(x,y) \in H^1(\Omega)$

Weak

$$\int_{\Omega} -(\nabla^2 u)v - \lambda e^u v = 0$$

$$\Leftrightarrow -\int_{\partial\Omega} \cancel{\nu} \nabla u \cdot \hat{n} + \int_{\Omega} \nabla u \cdot \nabla v - \lambda e^u v = 0$$

$$F(u) = \boxed{\int_{\Omega} \nabla u \cdot \nabla v - \lambda e^u v = 0} \quad \begin{array}{l} \text{for all} \\ v \in H_0^1(\Omega) \end{array}$$

Firedrake:

$$F = (\text{dot}(\text{grad}(u), \text{grad}(v)) - \text{lam} * \exp(u) * v) * dx$$

Newton method

$u^{(0)}$ initial guess of solution

$$F(u^{(k+1)}) = 0 \quad \leftarrow \text{want this}$$

$$F(u) \approx \underbrace{F(u^{(k)}) + J_F(u^{(k)})(u - u^{(k)})}$$

↑ linear approx. around $u^{(k)}$

Newton step equation:

Firedrake

computes this derivative (Jacobian)

update:

for you

$$\underbrace{J_F(u^{(k)}) S}_{\text{circled}} = -F(u^{(k)})$$

basically: $As = b$

$$u^{(k+1)} = u^{(k)} + S$$

need to know: for nonlinear PDEs,

- set up weak form as usual

$$F = \dots$$

- call solve ($F == 0$, u , bcs, ...) as usual ...

- but use these or similar:

'snes-type': 'newtonls',

'snes-converged-reason': None,

'snes-linesearch-type': 'basic',

'snes-rtol': $1e-8$,

Firedrake defaults

- PETSc's SNES component will manage the Newton iteration, and you will get feedback on its success (esp. `SNES_converged_reason`)
- what you want to see is a small number of Newton iterations, insensitive to resolution
- for nonlinear problems it does matter what is your initial iterate in u

demo!
bratu.py