FE Seminar

Thurs 1 Feb 2024

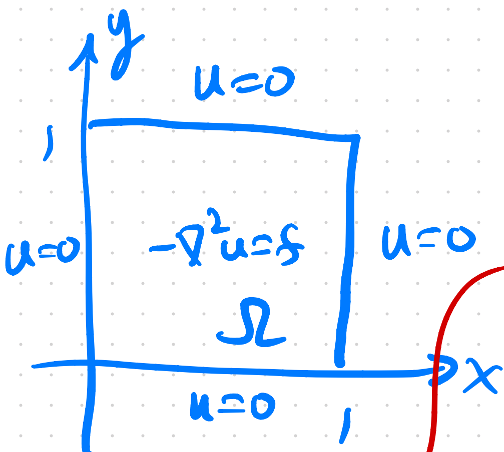
A. verification example

py/1feb/{draft.py $\xrightarrow{\text{edits}}$ exact.py}

B. FE assembly: how does it work? } basics only

C. in what sense is the FE solution close to the PDE solution?

A. problem solved by exact.py



Choose

$$u(x,y) = x(1-x)\sin(\pi y)$$

(observe $u|_{\partial\Omega} = 0$ by construction)

compute

$$f(x,y) = -\nabla^2 u$$
$$= -u_{xx} - u_{yy}$$
$$= -(-2)\sin(\pi y) - \left(x(1-x)(-\pi^2)\sin(\pi y)\right)$$
$$= \left(2 + \pi^2 x(1-x)\right)\sin(\pi y)$$

this cheat is called "method of manufactured solutions" = MMS

B. FE assembly

- recall derivation, integration by parts:

$$-\nabla^2 u = f \longrightarrow \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \quad \text{for all } v \in V = H_0^1$$

strong form PDE

weak form needed by Firedrake:

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F = (dot(grad(u), grad(v))
     - f*v) *dx
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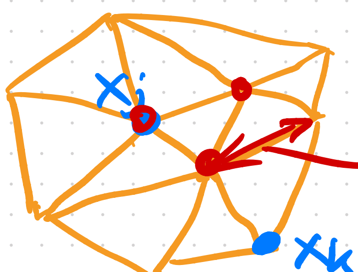
- inside Firedrake, once you choose mesh and etc., further transformed:

$$\longrightarrow A_h u_h = b_h \quad \Big\} \text{ invertible linear system}$$

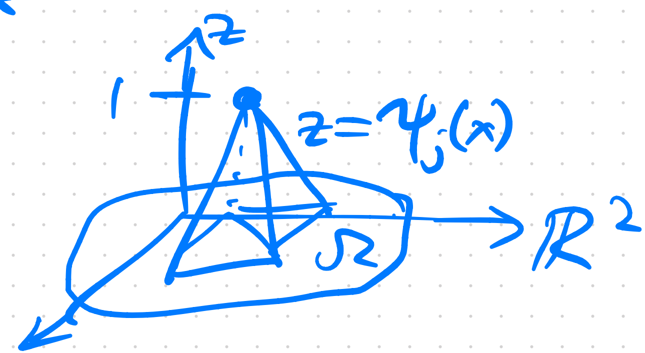
- even roughly, how is that linear system assembled?

def: for a given mesh and function space V_h, the hat function for each mesh node x_j is the element ψ_j of V_h so that

$$\psi_j(x_k) = \begin{cases} 1, & k=j \\ 0, & \text{otherwise} \end{cases}$$



$h = $ (mesh diameter) $=$ (spacing)



Claim: ① the set of all hat functions is a basis for V_h prove via lin. alg.

② to solve

$$\int_\Omega \nabla u_h \cdot \nabla v_h \overset{(*)}{=} \int_\Omega f v_h \quad \text{for all } v_h \in V_h$$

for the solution $u_h \in V_h$, it suffices to

(i) write $u_h(x) = \sum_{j=1}^{n} c_j \psi_j(x)$ $\{\psi_j\}$ is ~~a~~ basis of trial functions

(ii) apply $(*)$ for all $\psi_k(x)$ $\{\psi_k\}$ is basis of test functions

basic view of FE assembly:

to solve: $\int_\Omega \nabla u_h \cdot \nabla v_h = \int_\Omega f v_h$ for all $v_h \in V_h$,

do:

① form $A_h \in \mathbb{R}^{n\times n}$, $b_h \in \mathbb{R}^n$ by

$$a_{ij} = \int_\Omega \nabla \varphi_i \cdot \nabla \varphi_j, \quad b_i = \int_\Omega f \varphi_i$$

② solve by (e.g.) Gauss elimination

$$A_h c = b_h$$

③ the solution is $u_h(x) = \sum_{j=1}^{n} c_j \varphi_j(x)$

Comments:

- resulting A is sparse

  because: supports of hats functions usually don't overlap

- in fact, assembly is done element-wise, i.e. with "element stiffness matrices"

C. how close is u_h to u?

recall:

u solves $\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \quad \forall v \in H_0^1(\Omega) = V$

u_h solves $\int_\Omega \nabla u_h \cdot \nabla v_h = \int_\Omega f v_h \quad \forall v_h \in V_h$ ($= P_1$ for example)

Q: how big is

$$u - u_h\ ?$$

(note we saw this for exact.py)

def: • the **energy norm** (associated to the Poisson equation) is

$$\|w\| = \left(\int_\Omega |\nabla w|^2 \right)^{1/2} = \left(\int_\Omega \nabla w \cdot \nabla w \right)^{1/2}$$

• the **bilinear form** is

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v$$

• the **source functional** is $\ell(v) = \int_\Omega f v$

observe:

① $a(u,u) = \|u\|^2$

② the weak form is "$a(u,v) = \ell(v) \quad \forall v \in V$"

Cea's lemma:

$$\|u-u_h\| = \min_{v_h \in V_h} \|u-v_h\|$$

interpretation:

proof part 1

$a(u-u_n, v_n) = 0$

proof part 2

- So what? does Cea's lemma say anything quantitative?