Functions defined in pieces: simple, step, and piecewise-linear

a calculation for week 6

Ed Bueler

UAF Math 617 Functional Analysis

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Outline

1. measurable subsets of $\mathbb{R}^1$, and Lebesgue measure
2. simple functions
3. step functions ($=$ piecewise-constant on meshes)
4. piecewise-linear functions
5. piecewise functions on triangles (*first look at finite elements*)
6. theory from the book

- in Chapter 3 our textbook defines *measurable subset* (of $\mathbb{R}^n$), and then *simple function* (from $\mathbb{R}^n$ to $\mathbb{R}$)
- I will give the definitions for $\mathbb{R}^1$ in these slides

Definition

- for real numbers $a \le b$, a *finite interval* $I \subset \mathbb{R}^1$ is either

$$(a, b) \text{ or } (a, b] \text{ or } [a, b) \text{ or } [a, b]$$

- the *length* of a (finite) interval is

$$m(I) = b - a < \infty$$

- singleton sets, and the empty set, are finite intervals with length zero
  - $\{a\} = [a, a]$ and $\emptyset = (a, a)$

Definition

for any subset $A \subset \mathbb{R}^1$, the *outer measure* is

$$m^*(A) = \inf \left\{ \sum_{k=1}^{\infty} m(I_k) \,:\, A \subset \bigcup_{k=1}^{\infty} I_k \right\}$$

where the infimum is over countable unions of (finite) intervals I_k covering A

- $m^*(A)$ is defined for *any* subset $A \subset \mathbb{R}^1$
- thus $m^* : 2^{(\mathbb{R}^1)} \to [0, +\infty]$
  - the *power set* of a set S is $2^S = \{A \,:\, A \subset S\}$
- $m^*(A) = +\infty$ is possible

outer measure: examples

Ex 1 if I is any interval then $m^*(I)$ is the length of I

- $m^*(\mathbb{R}) = +\infty$, and likewise for half-infinite intervals like $[a, +\infty)$

Ex 2 if A is a finite or countable set then $m^*(A) = 0$

Proof.

□

Ex 3 $m^*(\mathbb{Q}) = 0$

Ex 4 $m^*([0,1] \setminus \mathbb{Q}) = 1$

Ex 5 $m^*(C) = 0$ where $C \subset [0,1]$ is the *Cantor middle-thirds set* [*picture*]

Definition

a subset $A \subset \mathbb{R}^1$ has *measure zero* if $m^*(A) = 0$

Lemma

$A \subset \mathbb{R}^1$ has measure zero if and only if for every $\epsilon > 0$ there is a countable covering of A by intervals of total length less than ϵ:

$$\sum_{k=1}^{\infty} m(I_k) < \epsilon \qquad \text{where } A \subset \bigcup_{k=1}^{\infty} I_k$$

Proof. Just think through the definition of $m^*(A) = 0$. □

- every subset of a measure zero set has measure zero
- every finite or countable set has measure zero
- the Cantor set $C \subset [0, 1]$, though it has uncountable cardinality, has measure zero

Definition

given two sets A, B, their *symmetric difference* is the set

$$S(A,B) = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$$

picture:

Definition

section 3.2 defines three collections of subsets of $\mathbb{R}^1$:

- $\mathcal{E}$ is the collection of finite, disjoint unions of (finite) intervals
- $\mathcal{M}_\mathcal{F}$ is the collection of all $A \subset \mathbb{R}^1$ such that

$$m^*(S(A_k, A)) \to 0 \text{ as } k \to \infty$$

  for some sequence $A_k \in \mathcal{E}$
- $\mathcal{M}$ is the collection of finite and countable unions of sets from $\mathcal{M}_\mathcal{F}$

$$\mathcal{E} \subset \mathcal{M}_\mathcal{F} \subset \mathcal{M} \subset 2^{(\mathbb{R}^1)}$$

Lebesgue measurable subsets of $\mathbb{R}^1$

Definition

$\mathcal{M}$ is the collection of *Lebesgue measurable sets*

- every interval is measurable: $I \in \mathcal{M}$
- every finite union of intervals is measurable: $\mathcal{E} \subset \mathcal{M}$
- every set of measure zero is measurable
- every measurable set can be well-approximated by a countable, disjoint union of intervals:

$$A \in \mathcal{M} \iff \forall \epsilon > 0 \, \exists \, B = \bigcup_{k=1}^{\infty} I_k \text{ so that } m^*(S(A,B)) < \epsilon$$

- there exist $A \subset \mathbb{R}^1$ that are not Lebesgue measurable:

$$\mathcal{M} \subsetneq 2^{(\mathbb{R}^1)}$$

- I will be covering this Chapter 3 material somewhat more carefully in the next few days

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Lemma

$G \subset \mathbb{R}^1$ is open if and only if it is a countable, disjoint union of open intervals:

$$G = \bigcup_{k=1}^{\infty} (a_k, b_k) \qquad \textit{where } a_k, b_k \in [-\infty, +\infty]$$

- thus $m^*(G) = \sum_k b_k - a_k$ for open sets G

Lemma

a measurable set can be squeezed between a closed and an open set:

$$A \in \mathcal{M} \qquad \iff \qquad \begin{array}{c} \forall \epsilon > 0 \, \exists E \text{ closed } \exists G \text{ open s.t.} \\ E \subset A \subset G \text{ and } m^*(G \setminus E) < \epsilon \end{array}$$

picture:

but what is “Lebesgue measure”?

Definition

Lebesgue measure is the function $m : \mathcal{M} \to [0, +\infty]$ defined as the outer measure restricted to the measurable sets:

$$\begin{aligned} m : \mathcal{M} &\to [0, +\infty] \\ A &\mapsto m^*(A) \end{aligned}$$

- what's the key property of a measure?

Lemma

m is countably additive*, that is, if $A_k \in \mathcal{M}$ are pairwise disjoint ($A_j \cap A_k = \emptyset$ for any $j \neq k$) then*

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} m(A_k)$$

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simple functions

Definition

the *indicator function* for the set $A \subset \mathbb{R}^1$ is

$$\mathbb{1}_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

- also denoted $\chi_A(x)$

Definition

a *simple function*, $s : \mathbb{R}^1 \to \mathbb{R}$, is a finite linear combination of indicator functions of measurable sets $E_k \in \mathcal{M}$, using coefficients $c_k \in \mathbb{R}$:

$$s(x) = \sum_{k=1}^{N} c_k \mathbb{1}_{E_k}(x)$$

integrals of simple functions

- the integral of a simple function $s(x) = \sum_{k=1}^{N} c_k \mathbb{1}_{E_k}(x)$ is obvious?

Definition

if s is a measurable simple function then its *Lebesque integral* is

$$\int_{\mathbb{R}} s(x)\,dx = \sum_{k=1}^{N} c_k m(E_k),$$

assuming for simplicity that $\sum_k m(E_k) < \infty$

picture:

the fundamental idea of Lebesgue integrals

- idea: cut-up the *range* of $f : [a, b] \to \mathbb{R}$, so as to approximate by a simple function, and integrate that simple function
  - f has to be *measurable* $\longleftarrow$ we will define!
- the range is cut into $y_0 < y_1 < \cdots < y_N$ to define the simple function:

$$s(x) = \sum_{k=0}^{N-1} y_k \, \mathbb{1}_{f^{-1}([y_k, y_{k+1}))}(x)$$

  - $E_k = f^{-1}([y_k, y_{k+1}))$ are the measurable sets

Definition (nearly the right definition)

for a measurable $f : [a, b] \to \mathbb{R}$, the *Lebesgue integral* is

$$\int_a^b f(x)\, dx = \lim_{N\to\infty} \sum_{k=0}^{N-1} y_k \, m\left(f^{-1}([y_k, y_{k+1}))\right)$$

- the careful and correct definition first restricts to *nonnegative* functions, and the uses "sup" instead of the "$\lim_{N\to\infty}$"

the fundamental idea of Lebesgue integrals

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picture:

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step functions

- simple functions $s(x) = \sum_{k=1}^{N} c_k \mathbb{1}_{E_k}(x)$ are as complicated as the sets E_k
  - measurable sets E_k can be quite intricate, e.g. Cantor sets
  - message: simple functions aren't actually simple
- what if we restrict E_k to be intervals?
- given a 1D mesh $x_0 < x_1 < x_2 < \dots < x_n$, define the (finite) intervals to be $I_k = [x_{k-1}, x_k]$ or $[x_{k-1}, x_k)$ or $(x_{k-1}, x_k]$ or (x_{k-1}, x_k)

Definition (equivalent forms)

- given $x_0 < x_1 < x_2 < \dots < x_n$ and $c_k \in \mathbb{R}$ a *step function* is

$$q(x) = \sum_{k=1}^{N} c_k \mathbb{1}_{I_k}(x)$$

- a simple function is a *step function* if each E_k is a finite interval

Definition

given the mesh $x_0 < x_1 < x_2 < \cdots < x_n$ and $c_k \in \mathbb{R}$ a *step function* is

$$q(x) = \sum_{k=1}^{N} c_k \mathbb{1}_{I_k}(x),$$

where I_k is an interval of the mesh

picture:

- weirdly, our textbook gets this definition wrong!
  - please correct it on page 67
  - the *statement* of Theorem 3.22 is correct

the two integrals (rough comparison)

- simple functions from meshing the range $\leftrightarrow$ Lebesgue integrals
- step functions from meshing the domain $\leftrightarrow$ Riemann integrals

integration concepts

- approximate $f : [a, b] \to \mathbb{R}$ by simple functions, integrate those using the Lebesgue measure, and take a limit to get the *Lebesgue integral*:

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{k=0}^{N-1} y_k\, m\left(f^{-1}([y_k, y_{k+1}))\right)$$

- approximate $f : [a, b] \to \mathbb{R}$ by step functions, integrate those, and take a limit to get the *Riemann integral*:

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{k=0}^{N-1} f(x_k^*)\,(x_k - x_{k-1})$$

Theorem

1. *for every* $f \in C([a,b])$ *and* $\epsilon > 0$ *there exists a step function* q *such that* $\|f - q\|_\infty < \infty$
2. *the simple functions are dense in* $L^p(X,\mu)$, $\|\cdot\|_p$, *for* $1 \le p \le \infty$
3. *the step functions are dense in* $L^p(E,m)$, $\|\cdot\|_p$, *for* $1 \le p < \infty$
   - *here* $E \subset \mathbb{R}^1$ *is measurable, e.g. any interval*
4. *the continuous functions* $C([a,b])$ *are dense in* $L^p([a,b],m)$ *for* $1 \le p < \infty$

- density of simple functions (item 2) in $L^p(X,\mu)$ is (essentially) the definition of the Lebesgue integral
  - this fact is completely general
- density of step functions $\implies$ density of simple functions
- 1 is a 19th century fact which follows from uniform continuity
  - I have already used this idea in proving the Riemann-Lebesgue lemma

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- restrict the (finite) intervals to be closed: $I_k = [x_{k-1}, x_k]$

Definition

- given a mesh $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ and coefficients $c_k, d_k \in \mathbb{R}$, a *continuous piecewise-linear function* on $[a, b]$ is

$$p(x) = \sum_{k=1}^{N} (c_k(x - x_{k-1}) + d_k)\, \mathbb{1}_{I_k}(x)$$

  if $c_{k-1}(x_k - x_{k-1}) + d_{k-1} \overset{*}{=} d_k$ for $k = 2, 3, \dots, N$

- a *discontinuous piecewise-linear "function"* is the same thing, but without requirement $*$

picture:

observation about piecewise functions

- I have already used piecewise functions many times in proofs and counter-examples
  - often these are “hat functions” or moving spikes etc.
  - the easiest constructable continuous functions are often piecewise-linear
  - the easiest constructable discontinuous functions are often piecewise-constant
- a piecewise-linear or piecewise-constant ($=$ step) function is defined using finitely-many real numbers, so it can be stored on a computer
- continuous piecewise-linear functions are foundational in applied and computational mathematics
  - generally: piecewise-polynomial functions
  - also finite Fourier series ($=$ trigonometric polynomials)
- simple and step functions are foundational in theory
  - in fact piecewise-constant functions are very useful in practice

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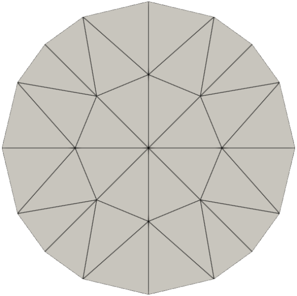
piecewise-linear and piecewise-constant functions on a polygon

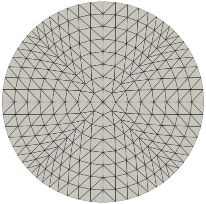
- modern *finite element* software is based (especially) on "step" and piecewise-linear functions on meshes in $\mathbb{R}^d$ (especially: $d = 1, 2, 3$)
- for example, Firedrake is a Python library that makes this easy
- example below: mesh a polygonal approximation of the disk, approximate $f(x, y) = 1 - x^2 - y^2$ by piecewise-linear, and put it in a file
- a separate program Paraview reads the file and generates the graphs

```
from firedrake import *

mesh = UnitDiskMesh(refinement_level=1)   # or ...=3)
x, y = SpatialCoordinate(mesh)

V = FunctionSpace(mesh, "CG", 1)  # or ..., "DG", 0)
f = Function(V, name="f").interpolate(1.0 - x**2 - y**2)

VTKFile("output.pvd").write(f)
```

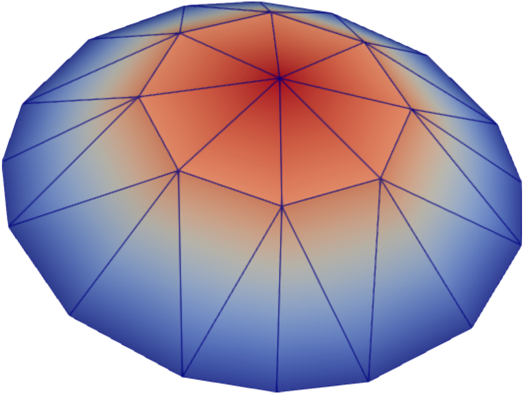
piecewise-linear (and piecewise-constant) on a 2D mesh



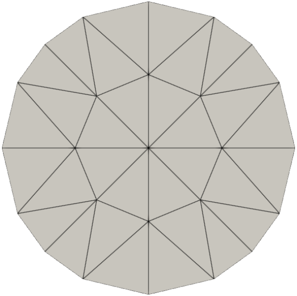
mesh of triangles



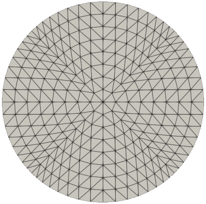
refined mesh



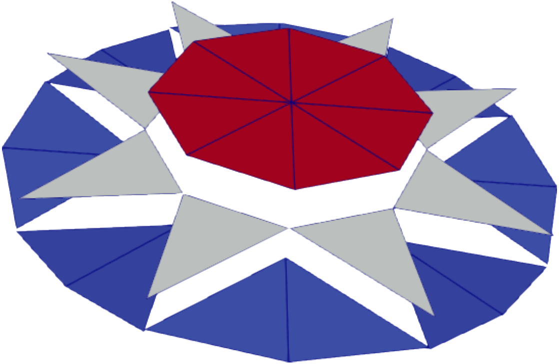
piecewise-linear interpolant of
$f(x, y) = 1 - x^2 - y^2$



mesh of triangles

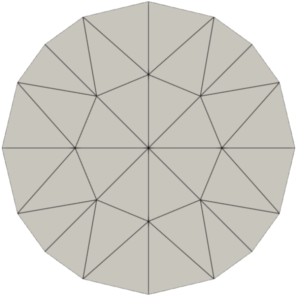


refined mesh

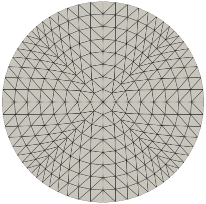


piecewise-constant interpolant of
$f(x,y) = 1 - x^2 - y^2$

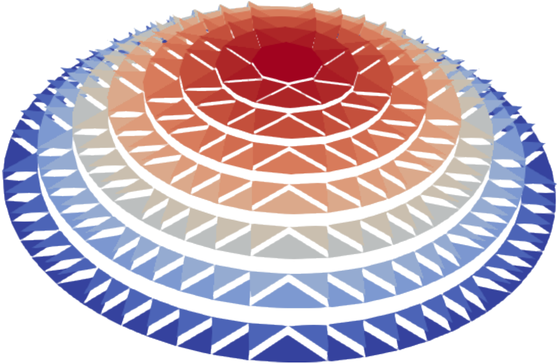
piecewise-linear (and piecewise-constant) on a 2D mesh



mesh of triangles



refined mesh



piecewise-constant interpolant of
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to define:

- *ring* of sets, and σ-*ring* of sets $\mathcal{R}$ (= *algebra* and σ-*algebra*)
- an abstract *measure* μ
- *Lebesgue outer measure* m^* on $\mathbb{R}^n$
- *Lebesgue measurable* subsets $\mathcal{M}$ of $\mathbb{R}^n$
- *Lebesgue measure* m
- *simple functions* and their integrals
- $f : \mathbb{R}^n \to \mathbb{R}$ is *measurable*
- *Lebesgue integral* $\int_E f\, dm$
- abstract/general: *measure space* $(X, \mathcal{R}, \mu)$, *measurable* $f : X \to \mathbb{R}$
- abstract/general: *Lebesgue integral* $\int_X f\, d\mu$
- general: $L^p(X, \mu)$ space

to prove ... or at least sketch the proof:

- $\mathcal{M}$ is a σ-ring and m^* is a measure on $\mathcal{M}$ (*which defines* $m = m^*|_{\mathcal{M}}$)
- Riemann integrable $\implies$ Lebesgue integrable
- monotone convergence theorem, Fatou's lemma, dominated convergence theorem
- $L^p(X,\mu), \|\cdot\|_p$ is a normed vector space
- $L^p(X,\mu)$ is complete (*thus a Banach space; Hilbert if* $p = 2$)
- Hölder's inequality: if $\frac{1}{p} + \frac{1}{q} = 1$ then $\|fg\|_1 \le \|f\|_p \|g\|_q$
- Minkowski inequality $=$ triangle inequality in $L^p(X,\mu)$
- step functions are dense in $L^p(X,\mu)$

- Assignment 4 is posted at bueler.github.io/fa