Fourier series of continuous functions

a calculation for weeks 4 & 5

Ed Bueler

UAF Math 617 Functional Analysis

Spring 2026

Outline

1. sines (and complex exponentials) are orthonormal
2. what did Fourier believe? (1822)
3. what did Dirichlet prove? (1829)
4. what is the full story for Fourier series on $C[-\pi, \pi]$? (1966)
5. what is the clean L^2 version of the story? ($\sim$1910)
6. theory from the book

the product-of-sines integral

- for $n \in \mathbb{N}$, $\sin(nx)$ is a *wave* on $[0, \pi]$
- suppose $m, n \in \mathbb{N}$ and integrate the product of sines:

$$\int_0^\pi \sin(mx)\sin(nx)\,dx =$$

$$= \begin{cases} \frac{\pi}{2}, & m = n \\ 0, & \text{otherwise} \end{cases}$$

$\cos(\theta \pm \psi) = \cos\theta\cos\psi \mp \sin\theta\sin\psi \quad \therefore \quad \sin\theta\sin\psi = \frac{1}{2}\left(\cos(\theta - \psi) - \cos(\theta + \psi)\right)$

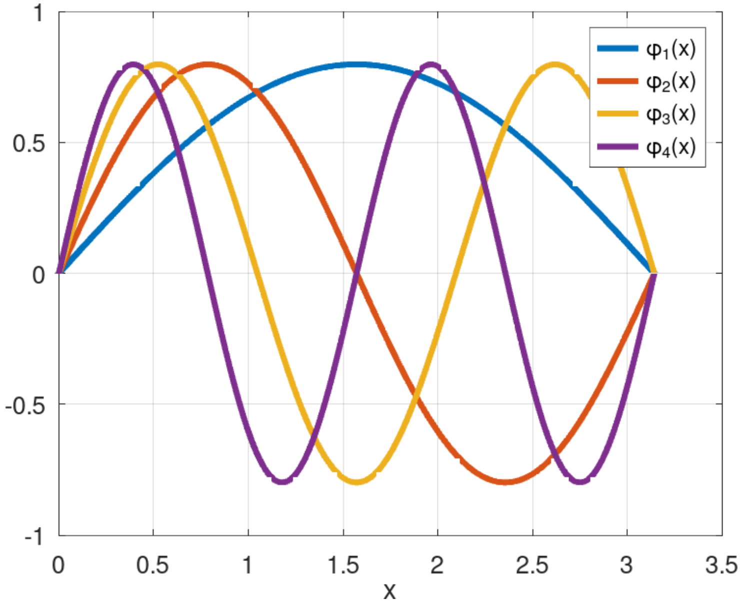
an infinite orthonormal set in $C[0,\pi]$

- fact from last slide: $\displaystyle \int_0^\pi \sin(mx)\sin(nx)\,dx = \frac{\pi}{2}\delta_{mn}$
- define

$$\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx) \qquad \text{and} \qquad S = \{\phi_n(x)\,:\,n\in\mathbb{N}\}$$

- then:
  1. S is orthonormal in the (real) inner product $\langle f, g\rangle = \int_0^\pi f(x)g(x)\,dx$:
  2. S is linearly independent:
- *questions:* what is $\text{span}(S)$? what is $\overline{\text{span}(S)}$?

an infinite orthonormal set in $C[0, \pi]$



a complex-valued orthonormal set in $C[-\pi,\pi]$

- *easier* integral:

$$\int_{-\pi}^{\pi} e^{imx} e^{-inx}\,dx = \int_{-\pi}^{\pi} e^{i(m-n)x}\,dx = \qquad\qquad = 2\pi\delta_{mn}$$

- define

$$\psi_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx} \qquad \text{and} \qquad E = \{\psi_n(x)\,:\,n\in\mathbb{Z}\}$$

- recall *complex* inner product (*sesquilinear* :-)), now on $C[-\pi,\pi]$:

$$\langle f,g\rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx$$

- then:
  1. E is orthonormal
  2. E is linearly-independent
- same big *question:* what is $\overline{\mathrm{span}(E)}$?

before we move on ...

basic facts about $e^{i\theta}$:

- $e^{i\theta} = \cos\theta + i\sin\theta$
  - derive this from $e^z = 1 + z + \frac{z^2}{2!} + \dots + \frac{n^2}{n!} + \dots$
  - also recall Taylor series for $\cos$ and $\sin$
- $\cos\theta = \dfrac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right), \quad \sin\theta = \dfrac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

basic facts about even and odd Fourier series:

- if f is even then

$$\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = \int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = 2\int_0^{\pi} f(x)\cos(nx)\,dx$$

- if f is odd then

$$\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx = i\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = 2i\int_0^{\pi} f(x)\sin(nx)\,dx$$

Outline

Fourier's assertion

claim (Fourier, *Théorie Analytique de la Chaleur*, 1822)

if $f \in C[0,\pi]$, and if we compute these coefficients

$$c_n = \langle f, \phi_n \rangle = \sqrt{\frac{2}{\pi}} \int_0^\pi f(x) \sin(nx)\, dx$$

for $n \in \mathbb{N}$, then

$$f(x) = \sum_{n=1}^{\infty} c_n\, \phi_n(x) = \sqrt{\frac{2}{\pi}} \sum_{n=1}^{\infty} c_n \sin(nx)$$

- the result is the *Fourier sine series* (or *expansion*) of f
- actually Fourier claimed this for discontinuous functions too
  - observe that c_n is an *integral*, so only features of f that change the integrals can affect the expansion
  - Fourier, and everybody else, agreed that the claim could not be exactly true at the points of discontinuity of $f(x)$

complex version is easier

- Fourier used real numbers (to my knowledge)
- we may replace

$$C[0,\pi] \to C[-\pi,\pi] \qquad \text{and} \qquad \sin(nx) \to e^{inx},$$

and use the complex (sesquilinear) inner product

Fourier's claim for complex exponentials

if $f \in C[-\pi,\pi]$, complex-valued, and if

$$c_n = \langle f, \psi_n \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

for $n \in \mathbb{Z}$, then

$$f(x) = \sum_{n\in\mathbb{Z}} c_n\, \psi_n(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} c_n e^{inx}$$

- this is the *Fourier series* of f, or the *complex Fourier series*
- for formulas from different references, be careful where the 2π goes

example 1: $f(x) = x$

- consider the Fourier sine series for $f(x) = x$, using the real orthonormal set $\{\phi_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)\} \subset C[0,\pi]$
- integrate to get coefficients:

$$c_n = \langle f, \phi_n \rangle = \sqrt{\frac{2}{\pi}} \int_0^\pi x \sin(nx)\,dx =$$

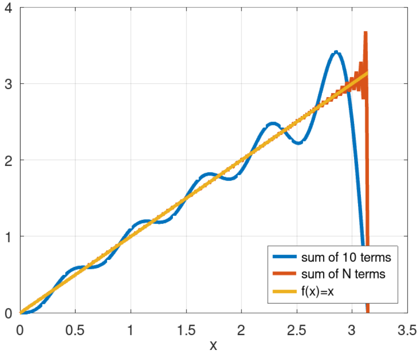
$$= \frac{\sqrt{2\pi}(-1)^{n-1}}{n}$$

example 1: $f(x) = x$

- MATLAB:

```
f = @(x) x;
x = 0:pi/300:pi;   N = 150;   nn = 1:N;
c = sqrt(2 * pi) * (-1).^(nn - 1) ./ nn;
sN = zeros(size(x));
for n=1:N
    sN = sN + c(n) * sqrt(2/pi) * sin(n*x);   end
plot(x, f(x), x, sN)
```

- result (works . . . but *Gibbs effect*!):



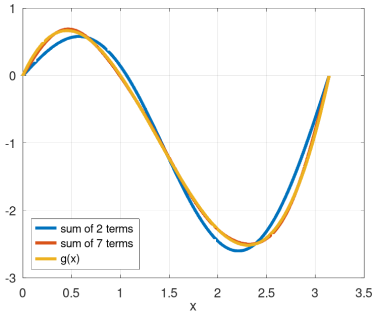
example 2: $g(x)$ cubic

- note that if $f(x) = x$ is *extended periodically* to the real line then the result is *not* continuous

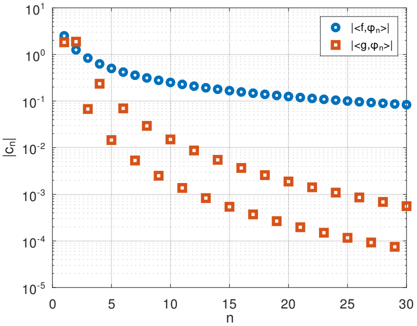
---

- instead consider $g(x) = x(x-1)(x-\pi)$ on the interval $[0, \pi]$
- now the periodic extension *is* continuous
- coefficients, done numerically: $$c_n = \langle g, \phi_n \rangle = \sqrt{\frac{2}{\pi}} \int_0^\pi g(x) \sin(nx)\, dx$$

MATLAB result:



coefficient decay:



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approximating f

- again consider complex Fourier series for $f \in C[-\pi, \pi]$, namely

$$c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx, \qquad f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} c_n e^{inx}$$

- combine into one formula and exchange limit processes:

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} \left(\frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(y) e^{-iny}\, dy \right) e^{inx} \\ &= \int_{-\pi}^{\pi} \left(\frac{1}{2\pi} \sum_{n\in\mathbb{Z}} e^{-iny} e^{inx} \right) f(y)\, dy = \int_{-\pi}^{\pi} \left(\frac{1}{2\pi} \sum_{n\in\mathbb{Z}} e^{in(x-y)} \right) f(y)\, dy \end{aligned}$$

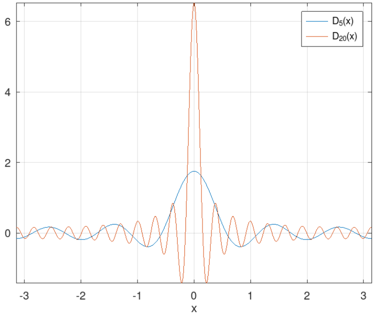
  - exchange of integral and sum is justified if $\sum |c_n| < \infty$

- if the inner sum is approximated by a partial sum, then we are approximating f by integrating it against a kernel

Dirichlet's kernel

Definition (Dirichlet's kernel, 1829)

$$D_m(x) = \frac{1}{2\pi} \sum_{n=-m}^{m} e^{inx} \overset{\text{why?}}{=} \frac{1}{2\pi} + \frac{1}{\pi} \sum_{n=1}^{m} \cos(nx)$$



Fourier's claim has become:

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} c_n e^{inx}$$
$$= \lim_{m\to\infty} \int_{-\pi}^{\pi} D_m(x-y)\, f(y)\, dy$$

Definition (Dirichlet's kernel)

$$D_m(x) = \frac{1}{2\pi} \sum_{n=-m}^{m} e^{inx}$$

Lemma (properties of the kernel)

1. *D_m is continuous,*
2. $\displaystyle\int_{-\pi}^{\pi} D_m(x)\,dx = 1,$
3. $D_m(0) = \frac{m+1/2}{\pi}$*, thus* $D_m(0) \to \infty$ *as* $m \to \infty$*, and*
4. $\displaystyle D_m(x) = \frac{1}{2\pi} \frac{\sin((m+1/2)x)}{\sin(x/2)}$ *for* $x \neq 0$*.*

Proof. Properties 1, 2, 3 are easy from the definition of D_m. Property 4 is on the next slide. □

Dirichlet's kernel as a sine ratio

Property ④: $D_m(x) = \dfrac{1}{2\pi} \dfrac{\sin((m+1/2)x)}{\sin(x/2)}$

Proof. Apply knowledge of geometric series, then use $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$:

$$\begin{aligned}
\sum_{n=-m}^{m} e^{inx} &= \sum_{n=-m}^{m} \left(e^{ix}\right)^n = e^{-imx} \sum_{n=0}^{2m} \left(e^{ix}\right)^n \\
&= e^{-imx} \frac{1-(e^{ix})^{2m+1}}{1-e^{ix}} = \frac{e^{-imx} - e^{i(m+1)x}}{1-e^{ix}} \\
&= \frac{e^{-i(m+1/2)x} - e^{i(m+1/2)x}}{e^{-i(x/2)} - e^{i(x/2)}} \\
&= \frac{\sin((m+1/2)x)}{\sin(x/2)}
\end{aligned}$$

The above calculation applies for $x \neq 0$. □

Dirichlet's theorem (1829)

Theorem

If $f \in C[-\pi, \pi]$ *is periodic, and if* $f'(x)$ *exists at* $x \in [-\pi, \pi]$*, then*

$$\lim_{m\to\infty} \int_{-\pi}^{\pi} D_m(x-y)\, f(y)\, dy = f(x),$$

and thus Fourier's claim is true:

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} c_n e^{inx} \qquad \text{for } c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx$$

- The proof is based on the *Riemann-Lebesgue lemma*. This version only requires the Riemann integral, and Dirichlet must have known it. I will prove it after using it.

Lemma (Riemann-Lebesgue for continuous functions)

If $g \in C[a, b]$ *then* $\displaystyle\lim_{n\to\infty} \int_a^b g(x) e^{inx}\, dx = 0.$

Dirichlet's theorem (1829)

Proof. By the periodicity of D_m and f, and since D_m is even, use $y = x + \xi$ to show

$$\int_{-\pi}^{\pi} D_m(x-y)\, f(y)\, dy = \int_{-\pi}^{\pi} D_m(\xi)\, f(x+\xi)\, d\xi$$

By replacing $f(x+\xi)$ with $\tilde{f}(\xi)$, we may assume $x = 0$ and $f'(0)$ exists. Next we calculate using the proven properties of the kernel:

$$\begin{aligned} \int_{-\pi}^{\pi} D_m(\xi)\, f(\xi)\, d\xi &= \int_{-\pi}^{\pi} D_m(\xi)\, \left(f(\xi) - f(0)\right)\, d\xi + \int_{-\pi}^{\pi} D_m(\xi)\, f(0)\, d\xi \\ &= f(0) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(f(\xi) - f(0)\right) \frac{\sin((m+1/2)\xi)}{\sin(\xi/2)}\, d\xi \end{aligned}$$

However, $f'(0)$ exists, so by L'Hopital's rule,

$$\lim_{\xi \to 0} \frac{f(\xi) - f(0)}{\sin(\xi/2)} = \lim_{\xi \to 0} \frac{f(\xi) - f(0)}{\xi} \frac{\xi}{\sin(\xi/2)} = 2f'(0)$$

Dirichlet's theorem (1829)

Proof continued. So now we know that this function is continuous (removable discontinuity):

$$h(x) = \begin{cases} \frac{f(x)-f(0)}{\sin(x/2)}, & x \neq 0 \\ 2f'(0), & x = 0 \end{cases}$$

Thus by the Riemann-Lebesgue lemma:

$$\begin{aligned} \lim_{m\to\infty} \int_{-\pi}^{\pi} D_m(\xi)\, f(\xi)\, d\xi &= f(0) + \frac{1}{2\pi} \lim_{m\to\infty} \int_{-\pi}^{\pi} h(\xi) \sin((m+1/2)\xi)\, d\xi \\ &= f(0) + 0 = f(0). \end{aligned}$$

□

Lemma

If $g \in C[a,b]$ then $\lim_{n\to\infty} \int_a^b g(x) e^{inx}\, dx = 0$.

Proof. Let $\epsilon > 0$. Since g is continuous on the compact set $[a,b]$, it is uniformly continuous, so there is $\delta > 0$ such that $|x - y| < \delta \implies |g(x) - g(y)| < \epsilon/(2(b-a))$. In particular we can define a mesh of points $\{x_i\}_{i=0}^M$, with spacing less than δ, and x_i^* at the midpoints of each $[x_{i-1}, x_i]$, so that

$$\tilde g(x) = \sum_{i=1}^M g(x_i^*) \mathbb{1}_{[x_{i-1},x_i)}(x)$$

is a uniform approximation of g: $\quad \|\tilde g - g\|_\infty = \sup_{x\in[a,b]} |\tilde g(x) - g(x)| < \dfrac{\epsilon}{2(b-a)}$

picture:

Riemann-Lebesgue lemma for continuous functions

Proof continued. Now

$$\begin{aligned}\left|\int_a^b g(x)e^{inx}\,dx\right| &\le \left|\int_a^b (g(x)-\tilde g(x))e^{inx}\,dx\right| + \left|\int_a^b \tilde g(x)e^{inx}\,dx\right| \\ &\le \|g-\tilde g\|_\infty \int_a^b |e^{inx}|\,dx + \left|\int_a^b \sum_{i=1}^M g(x_i^*)\mathbb{1}_{[x_{i-1},x_i)}(x)e^{inx}\,dx\right| \\ &\le \frac{\epsilon}{2(b-a)}(b-a) + \sum_{i=1}^M |g(x_i^*)| \left|\int_{x_{i-1}}^{x_i} e^{inx}\,dx\right|\end{aligned}$$

But each of the final integrals is bounded by the same constant:

$$\left|\int_{x_{i-1}}^{x_i} e^{inx}\,dx\right| = \left|\frac{e^{inx_i}-e^{inx_{i-1}}}{in}\right| \le \frac{2}{n}$$

Choose N so that $(2\|g\|_\infty M)/N < \epsilon/2$. If $n \ge N$ then

$$\left|\int_a^b g(x)e^{inx}\,dx\right| \le \frac{\epsilon}{2} + \frac{2}{n}\sum_{i=1}^M |g(x_i^*)| \le \frac{\epsilon}{2} + \frac{2}{N}\sum_{i=1}^M \|g\|_\infty < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \square$$

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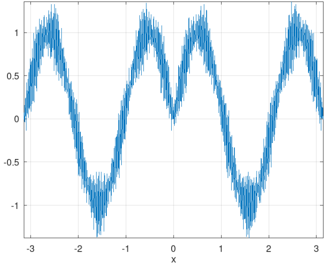
does the Fourier series converge for any $f \in C[-\pi,\pi]$?

- in 1873 Du Bois-Reymond constructed a continuous function for which the Fourier series does not converge at a point
    - found by google search: $$f(x) = \sum_{k=1}^\infty \frac{1}{k^2} \sin\left(\frac{(2^{k^3}+1)|x|}{2}\right)$$
    - the function is even and continuous,[1] so its Fourier series is a cosine series
    - it can be shown that the Fourier partial sums are unbounded in a neighborhood of $x=0$



- in the early 1900s, further examples were constructed where convergence failed on infinite sets of points
- to state the full situation we need to define *measure zero* ...

[1] use Weierstrass M-test

Definition

- a set $E \subset \mathbb{R}$ has *(Lebesgue) measure zero* if for any $\epsilon > 0$ there exists a countable list of open intervals $I_n = (a_n, b_n)$ so that

$$E \subset \bigcup_{n\in\mathbb{N}} I_n \quad \text{and} \quad \sum_{n\in\mathbb{N}} m(I_n) = \sum_{n\in\mathbb{N}} b_n - a_n < \epsilon$$

- if a proposition holds except on a set E which has measure zero then we say that it holds *almost everywhere*

- note that any open subset of $\mathbb{R}$ is a countable union of open intervals
- more on Lebesgue measure m, and measure zero, in Chapter 3

the full story for Fourier series of continuous functions

Theorem (Carleson, 1966)

if $f \in C[-\pi,\pi]$, or even $f \in L^p[-\pi,\pi]$, then the Fourier series of f converges pointwise almost everywhere:

$$\lim_{m\to\infty} \sum_{n=-m}^{m} c_n e^{inx} = f(x) \quad \textit{except on } x \in E \textit{ of measure zero}$$

Theorem (Katznelson, 1966)

for any set $E \subset [-\pi,\pi]$ of measure zero there is a continuous periodic function $f \in C[-\pi,\pi]$ whose Fourier series diverges at all points of E

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orthonormal sequences in inner product spaces

- the abstract ideas of Hilbert and Riesz, circa 1910, are fundamentally *simpler* than all of the previous material in these slides
- consider an abstract inner product vector space $V, \langle \cdot, \cdot \rangle$

Definition

- a pair of vectors $v, w \in V$ are *orthogonal* if $\langle v, w \rangle = 0$
- a set S is *orthogonal* if $v, w \in S$ and $v \neq w$ implies v, w are orthogonal
- $(v_k)_{k=1}^{\infty} \subset V$ is an *orthonormal sequence* (*ON sequence*) if v_k, v_l are orthogonal for $k \neq l$ and if $\|v_k\|^2 = \langle v_k, v_k \rangle = 1$

Lemma

- *an orthogonal set is linearly-independent*
- *an ON sequence is linearly-independent*

examples of orthonormal sequences

- $\left\{\sqrt{\frac{2}{\pi}}\sin(nx)\right\}_{n\in\mathbb{N}}$ is a (real) ON sequence on $[0,\pi]$
- $\left\{\frac{1}{\sqrt{2\pi}}e^{inx}\right\}_{n\in\mathbb{Z}}$ is an ON sequence on $[-\pi,\pi]$
- $\left\{\frac{1}{\sqrt{2\pi}}e^{inx}\right\}_{n\in\mathbb{N}}$ is an ON sequence on $[-\pi,\pi]$
- $\left\{\sqrt{\frac{2}{\pi}}\cos(nx)\right\}_{n\in\mathbb{N}}$ is a (real) ON sequence on $[0,\pi]$
- $\left\{\sqrt{\frac{1}{\pi}}\sin(nx)\right\}_{n\in\mathbb{N}}$ is a (real) ON sequence on $[-\pi,\pi]$
- the Legendre polynomials[2] form an ON sequence

[2] with a different normalization these are: $1, x, \frac{1}{2}(3x^2-1), \frac{1}{2}(5x^3-3x), \dots$

examples of orthonormal sequences

- $\left\{\sqrt{\frac{2}{\pi}}\sin(nx)\right\}_{n\in\mathbb{N}}$ is a (real) ON sequence on $[0,\pi]$ $\longleftarrow$ complete
- $\left\{\frac{1}{\sqrt{2\pi}}e^{inx}\right\}_{n\in\mathbb{Z}}$ is an ON sequence on $[-\pi,\pi]$ $\longleftarrow$ complete
- $\left\{\frac{1}{\sqrt{2\pi}}e^{inx}\right\}_{n\in\mathbb{N}}$ is an ON sequence on $[-\pi,\pi]$ $\longleftarrow$ *not* complete
- $\left\{\sqrt{\frac{2}{\pi}}\cos(nx)\right\}_{n\in\mathbb{N}}$ is a (real) ON sequence on $[0,\pi]$ $\longleftarrow$ *not* complete
- $\left\{\sqrt{\frac{1}{\pi}}\sin(nx)\right\}_{n\in\mathbb{N}}$ is a (real) ON sequence on $[-\pi,\pi]$ $\longleftarrow$ *not* complete
- the Legendre polynomials[2] form an ON sequence $\longleftarrow$ complete

[2] with a different normalization these are: $1, x, \frac{1}{2}(3x^2-1), \frac{1}{2}(5x^3-3x), \dots$

fundamental question about an orthonormal sequence

question

- given an ON sequence $S = (f_k)$ from an inner product space $(V, \langle\cdot,\cdot\rangle)$, is its span dense in V?:
$$\text{is } \overline{\operatorname{span} S} = V?$$

- this question becomes a definition ...

Definition

an ON sequence $S = (f_k)$ in an inner product space $(V, \langle\cdot,\cdot\rangle)$ is *complete* if for every $f \in V$ there exist coefficients c_k so that

$$f = \lim_{N\to\infty} \sum_{k=1}^{N} c_k f_k = \sum_{k=1}^{\infty} c_k f_k$$

Theorem 4.1

Suppose that $f = \sum_{k=1}^{\infty} c_k f_k$ for an ON sequence $S = (f_k)$ in an inner product space $V, \langle\cdot,\cdot\rangle$. Then

$$c_k = \langle f, f_k\rangle$$

- the proof is easy . . . we will do it

Definition

- for $f \in V$, the *Fourier coefficients* for an ON sequence $S = (f_k)$ are

$$c_k = \langle f, f_k\rangle$$

- the *Fourier series of* $f \in V$, for an ON sequence $S = (f_k)$, is

$$\sum_{k=1}^{\infty} \langle f, f_k\rangle\, f$$

how to prove an ON sequence is complete?

- the Fourier series is always in V, but that does *not* imply, by itself, that it equals f
- *if* the ON sequence is complete then f equals its Fourier series
- part (c) of the following theorem is the most practical way to show an ON sequence of functions on an interval is complete

Theorem 4.5

for an ON sequence $S = (f_k)_{k \in \mathbb{N}}$ in $L^2 = L^2([-\pi, \pi], m)$, the following are equivalent:

(a) S is a complete ON sequence

(b) for every $f \in L^2$ and $\epsilon > 0$ there is a finite linear combination $g = \sum_{k=1}^n d_k f_k$ so that $\|f - g\|_2 < \epsilon$

(c) if the Fourier coefficients of $f \in L^2$, for the ON sequence S, are zero then the function f is zero almost everywhere

- we will prove this Theorem

Fourier's claim is true *if* convergence is in L^2

Theorem 4.6 (rewritten in complex form)

The ON sequence $S = \left(\frac{1}{\sqrt{2\pi}} e^{inx}\right)_{n\in\mathbb{Z}}$ is complete in $L^2([-\pi,\pi],m)$. Therefore

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n e^{inx},$$

for these coefficients

$$c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx}\,dx,$$

in the sense that the partial sums converge to f in the norm $\|\cdot\|_2$

- this is the simplest interpretation of Fourier's claim
  - however, it does address continuous functions directly
- if $f(x)$ is real-valued we may write, for some real constants a_n, b_n,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

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theory in weeks 4 & 5: Chapter 4 in Saxe

- we will do Fourier series (Chapter 4) *before* Lebesgue integrals and measures (Chapter 3)

to define:

- $L^2([a,b],m)$ and $L^2(X,\mu)$ $\longleftarrow$ not fully defined till Chapter 3
- complete orthonormal sequence
- Fourier series and coefficients for any orthonormal sequence

to prove:

- Bessel's inequality
- Parseval's equality
- the Riesz-Fischer theorem
- Theorem 4.6: Fourier series converge in L^2

- Assignment 3 is posted at `bueler.github.io/fa`