Boundary value problems and integral operators

a calculation for weeks 2 & 3 (*version 3*)

Ed Bueler

UAF Math 617 Functional Analysis

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Outline

second-order BVP ("2-point problem")

suppose $\alpha, \beta \in \mathbb{R}$ and $\phi \in C([0,1])$ are given. solve for $u(x)$:

$$
\begin{aligned}
-u''(x) &= \phi(x) \qquad \text{for all } x \in (0,1) \\
u(0) &= \alpha \\
u(1) &= \beta
\end{aligned}
$$

- picture of data ϕ (left) and solution u (right):

- we will see this is *solvable by hand*, at least as an integral of the data $\phi(x)$, but this is mildly difficult, so we warm-up with an easier problem

easier: initial value problem

second-order IVP

suppose $\alpha, \beta \in \mathbb{R}$ and $\phi \in C([0,1])$ are given. solve for $u(x)$:

$$\begin{aligned} -u''(x) &= \phi(x) \qquad \text{for all } x \in (0,1) \\ u(0) &= \alpha \\ u'(0) &= \beta \end{aligned}$$

- picture:

solution of IVP

apply fundamental theorem of calculus twice:

$$-u''(x) = \phi(x)$$

solution of BVP

apply fundamental theorem of calculus twice: $\longleftarrow$ it will be more work

$$-u''(x) = \phi(x)$$

solution of BVP

result:

$$u(x) = \alpha + (\beta - \alpha)x + \int_0^1 k(x,t)\phi(t)\,dt$$

where k is a symmetric *kernel*:

$$k(x,t) = \begin{cases} t(1-x), & 0 \le t \le x \le 1 \\ x(1-t), & 0 \le x \le t \le 1 \end{cases} = \min\{x,t\} - xt$$

$k(x, t)$ for the BVP

picture of $k(x, t)$:

integral solution of BVP

- restrict to the $\alpha = \beta = 0$ special case

2-point BVP

the solution of
$$-u'' = \phi, \quad u(0) = u(1) = 0$$
is
$$u(x) = \int_0^1 k(x,t)\phi(t)\,dt \qquad \text{where } k(x,t) = \min\{x,t\} - xt$$

- u is the output of a linear *integral operator*

Outline

the integral operator

Definition

given a suitable *kernel* $k(x,t)$, the *integral operator* $L : C([0,1]) \to C([0,1])$ is

$$(L\phi)(x) = \int_0^1 k(x,t)\phi(t)\, dt$$

- L is linear:

- is L a matrix?

wait ... does $\phi \in C([0,1]) \implies L\phi \in C([0,1])$?

Definition

given a suitable *kernel* $k(x,t)$, the *integral operator* $L : C([0,1]) \to C([0,1])$ is

$$(L\phi)(x) = \int_0^1 k(x,t)\phi(t)\,dt$$

- what does "suitable" mean?
- is $L\phi$ even well-defined? is $L\phi \in C([0,1])$?
- for $\phi \in C([0,1])$, we see that $(L\phi)(x)$ is well-defined if $k(x,t)$ is (Riemann) integrable in t for every x
- what conditions on $k(x,t)$ guarantee that

$$\phi \in C([0,1]) \implies L\phi \in C([0,1])?$$

  - does $k(x,t)$ itself need to be continuous?

example: the antiderivative (as an integral operator)

Definition

for fixed $a \in [0, 1]$, an *antiderivative operator* $A : C([0, 1]) \to C([0, 1])$ is

$$(A\phi)(x) = \int_a^x \phi(t)\, dt$$

- the FTC shows this is an antiderivative:

- these integrals with variable limits give *Volterra integral operators*, whereas L on the previous slide is a *Fredholm integral operator*
  - look it up?

kernel of the antiderivative operator

- can the antiderivative operator $(A\phi)(x) = \int_0^x \phi(t)\,dt$ be put into the kernel form $(L\phi)(x) = \int_0^1 k(x,t)\phi(t)\,dt$?
- *yes*, and $k(x,t)$ is discontinuous (picture at right):

$$k(x,t) = \begin{cases} 1, & t \le x \\ 0, & t > x \end{cases} = \mathbb{1}_{[0,x]}(t)$$

- compute:

$$(L\phi)(x) = \int_0^1 k(x,t)\phi(t)\,dt = \int_0^1 \mathbb{1}_{[0,x]}(t)\,\phi(t)\,dt = \int_0^x \phi(t)\,dt = (A\phi)(x)$$

Lipschitz kernels

- regarding what conditions on $k(x,t)$ imply $L : C([0,1]) \to C([0,1])$, here is a sufficient condition

Definition

a function $k : [0,1] \times [0,1] \to \mathbb{R}$ is *Lipschitz in x* if there is $C \geq 0$ so that for all $x, y \in [0,1]$ and $t \in [0,1]$, $\quad |k(x,t) - k(y,t)| \leq C|x - y|$

Lemma

if $k(x,t)$ is Lipschitz in x, and if $(L\phi)(x) = \int_0^1 k(x,t)\phi(t)\,dt$ for $\phi \in C([0,1])$, then $L\phi \in C([0,1])$

- proof on next slide
- this is *not* a comprehensive theory of integral operators, but it gives me something to prove with elementary tools

Lipschitz kernels

Lemma

if $k(x,t)$ is Lipschitz in x, and if $(L\phi)(x) = \int_0^1 k(x,t)\phi(t)\,dt$ for $\phi \in C([0,1])$, then $L\phi \in C([0,1])$

Proof. Let $\phi \in C([0,1])$, and consider the function $L\phi$ at a point $x \in [0,1]$. Suppose $\epsilon > 0$. Let $\delta = \epsilon/(C\|\phi\|_\infty)$. If $y \in [0,1]$ and $|x-y| < \delta$ then by the triangle inequality for integrals, and the definition of Lipschitz,

$$\begin{aligned}
|(L\phi)(x) - (L\phi)(y)| &= \left| \int_0^1 k(x,t)\phi(t)\,dt - \int_0^1 k(y,t)\phi(t)\,dt \right| \\
&\le \int_0^1 |k(x,t) - k(y,t)|\,|\phi(t)|\,dt \le \int_0^1 C|x-y|\,|\phi(t)|\,dt \\
&\le \int_0^1 C|x-y|\,\|\phi\|_\infty\,dt = C\|\phi\|_\infty |x-y| < C\|\phi\|_\infty \delta = \epsilon \qquad \square
\end{aligned}$$

recall: solution of 2-point BVP

the solution of

$$-u'' = \phi, \quad u(0) = u(1) = 0$$

is $\quad u(x) = \int_0^1 k(x,t)\phi(t)\,dt \quad$ where

$$k(x,t) = \begin{cases} t(1-x), & t \le x \\ x(1-t), & x \le t \end{cases} \quad = \min\{x,t\} - xt$$

Corollary

since the above kernel is Lipschitz in x, with $C = 0.5$, it follows that $u \in C([0,1])$ if $\phi \in C([0,1])$

- but we already knew that *this* L sends $C([0,1])$ to itself …why?
- in fact: if $\phi \in C([0,1])$ then $u(x)$ has two derivatives

bounded kernel is not enough

Q. is integrability and boundedness of k sufficient for $L : C([0,1]) \to C([0,1])$?

A. *no.* there exist integrable, bounded, and discontinuous $k(x,t)$ so that $(L\phi)(x) = \int_0^1 k(x,t)\phi(t)\,dt$ is not continuous

example

$$k(x,t) = \begin{cases} 1, & x \le 0.5 \\ 0, & x > 0.5 \end{cases}$$

and $\phi(x) = 1$ gives

$$(L\phi)(x) =$$

- regarding necessary and sufficient conditions, I do not know!
- feel free to do research and try it yourself

pictures of three kernels

on t, x axes, sketch contour maps?:

$$k_1(x,t) = \begin{cases} t(1-x), & t \le x \\ x(1-t), & x \le t \end{cases} \qquad k_2(x,t) = \begin{cases} 1, & t \le x \\ 0, & t > x \end{cases} \qquad k_3(x,t) = \begin{cases} 1, & x \le 0.5 \\ 0, & x > 0.5 \end{cases}$$

Outline

$C([0,1]), \|\cdot\|_\infty$ is a normed vector space

- recall stuff from Chapter 1 of Saxe, *Beginning Functional Analysis*

Definition

for $f \in C([0,1])$, let $$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)| = \max_{x\in[0,1]} |f(x)|$$

- picture:

- if $V = C([0,1])$ then $(V, \|\cdot\|_\infty)$ is a normed vector space
  - there are four properties to check: (i), (ii), (iii), (iv)
  - recall: $d(f,g) = \|f-g\|_\infty$ makes (V,d) a metric space
- recall: convergence in $\|\cdot\|_\infty$ is *uniform* convergence

- now Chapter 2 of Saxe, *Beginning Functional Analysis*
- we assume $(V, \|\cdot\|)$ is a normed vector space
- recall: a subset $S \subset V$ is *open* if $\forall\, x \in S\ \exists\, r > 0$ so that $B_r(x) \subset S$
  - $B_r(x) = \{y \in V\ :\ \|y - x\| < r\}$ is an *open ball*

Definition

$\mathcal{C} = \{S_\lambda\}_{\lambda \in I}$ is an *open cover of* $E \subset V$ if $S_\lambda \subset V$ is open for each $\lambda \in I$ and $E \subset \bigcup_{\lambda \in I} S_\lambda$

Definition

- a subset $E \subset V$ is *compact* if every open cover of E has a finite subcover
- a subset $E \subset V$ is *sequentially compact* if every sequence in E contains a convergent subsequence

- fact: a finite set is compact

Theorem 2.4

$E \subset V$ is compact if and only if it is sequentially compact

the unit ball in $V = C([0,1])$ is not compact

Theorem 2.3

if $E \subset V$ is compact then it is closed

Theorem

the closed unit ball $B_1(0) = \{y \in V = C([0,1]) : \|y\|_\infty \le 1\}$ *is not compact*

Proof. The hat functions $\psi_n(x)$, $n \in \mathbb{N}$, drawn below, are continuous, and each has norm $\|\psi_n\|_\infty = 1$, thus $\psi_n \in B_1(0)$. Also, for any $n \ne m$, $\|\psi_n - \psi_m\|_\infty = 1$. Any subsequence (ψ_{n_k}) of the sequence (ψ_n) would not converge because again $\|\psi_{n_j} - \psi_{n_k}\|_\infty = 1$ for $j \ne k$. Thus $B_1(0)$ is not compact. □

- *Corollary.* if $E \subset V = C([0,1])$ contains a ball of positive radius $B_r(x) \subset E$, no matter how small $r > 0$ is, then E is not compact

why would we want sets to be compact?

Extreme Value Theorem (Weierstrass)

if $E \subset V$ is compact and $f : E \to \mathbb{R}$ is continuous then there exists a maximum and a minimum of f

- here $(V, \|\cdot\|)$ is any normed vector space
- the conclusion is that there are $c, C \in E$ so that

$$f(c) \le f(x) \le f(C)$$

  for all $x \in E$
- slightly more generally, one can consider continuous $f : M \to \mathbb{R}$ where (M, d) is any compact metric space
- compact subsets of $V = C([0, 1])$ are a major focus of Chapter 2

Outline

- assume $(V, \|\cdot\|)$ is a normed vector space
- the closed unit ball is $B = \{y \in V \,:\, \|y\| \leq 1\}$

Definition

- a linear operator $L : V \to V$ is *bounded* if there is a constant $C \geq 0$ so that
$$\|Lv\| \leq C\|v\| \qquad \text{for all } v \in V$$
- a linear operator $L : V \to V$ is *compact* if the image of the closed unit ball under L, that is, the set LB, is compact

- we will see that "bounded" is the same as "continuous" for *linear* operators

bounded integral operators on $C([0,1])$

Theorem

Let $V = C([0,1])$ with $\|\cdot\|_\infty$ as the norm. Consider the operator $L : V \to V$, $(L\phi)(x) = \int_0^1 k(x,t)\phi(t)\,dt$. If $k(x,t)$ is integrable and

$$\sup_{x\in[0,1]} \int_0^1 |k(x,t)|\,dt < \infty$$

then L is bounded.

Proof. Suppose $\phi \in V$. Then by the triangle inequality,

$$\begin{aligned}
\|L\phi\|_\infty &= \sup_{x\in[0,1]} |(L\phi)(x)| = \sup_{x\in[0,1]} \left| \int_0^1 k(x,t)\phi(t)\,dt \right| \\
&\le \sup_{x\in[0,1]} \int_0^1 |k(x,t)||\phi(t)|\,dt \le \sup_{x\in[0,1]} \int_0^1 |k(x,t)|\|\phi\|_\infty\,dt \\
&= \left(\sup_{x\in[0,1]} \int_0^1 |k(x,t)|\,dt \right) \|\phi\|_\infty
\end{aligned}$$

Thus $\|L\phi\|_\infty \le C\|\phi\|_\infty$ where $C = \sup_{x\in[0,1]} \int_0^1 |k(x,t)|\,dt$. □

new idea: equicontinuous sets of functions

Definition (see section 2.1)

a set $E \subset C([a,b])$ is *equicontinuous at* $x \in [a,b]$ if for each $\epsilon > 0$ there is $\delta > 0$ so that $f \in E$ and $y \in [a,b]$ and $|x-y| < \delta$ implies $|f(x) - f(y)| < \epsilon$

- this is about quantifiers!
- f continuous at $x \in [a,b]$:

$$\forall \epsilon > 0 \, \exists \delta > 0 \, \forall y \in [a,b] \quad |x-y| < \delta \implies |f(x) - f(y)| < \epsilon$$

- E equicontinuous at $x \in [a,b]$:

$$\forall \epsilon > 0 \, \exists \delta > 0 \, \forall f \in E \, \forall y \in [a,b] \quad |x-y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Definition

a set $E \subset C([a,b])$ is *uniformly equicontinuous* if for each $\epsilon > 0$ there is $\delta > 0$ so that $f \in E$ and $x, y \in [a,b]$ and $|x-y| < \delta$ implies $|f(x) - f(y)| < \epsilon$

- more on equicontinuity in Chapter 2 of Saxe, *Beginning Functional Analysis*

Ascoli-Arzelà Theorem

a set $E \subset C([a, b])$is compact, in the topology of the $\|\cdot\|_\infty$ norm, if and only if it is closed, bounded, and uniformly equicontinuous

- now we can return to the BVP!

Theorem

Consider $B_R(0) \subset V = C([0, 1])$. For $\phi \in B_R(0)$ the solution of

$$-u'' = \phi, \quad u(0) = u(1) = 0$$

is $u(x) = (L\phi)(x) = \int_0^1 k(x, t)\phi(t)\, dt$ where $k(x, t) = \min\{x, t\} - xt$. Then $u \in E$ where $E = LB_R(0)$ is a compact subset of V.

Proof. Show E is closed and bounded. Show E is uniformly equicontinuous by using the fact that $k(x, t)$ is Lipschitz in x. □

Outline

Definition

the *Dirac delta function* on $[0,1]$, centered at $t \in (0,1)$, satisfies

$$\delta_t(x) = \begin{cases} 0, & x \neq t \\ \text{positive}, & x = t, \end{cases}$$

and

$$\int_0^1 \delta_t(x)\,dx = 1$$

- picture:

- the above is a fake definition ...do not believe it
- the Dirac delta “function” is actually a positive *measure* $\delta_t(x)\,dx$

the kernel for the BVP comes from Dirac delta functions

Theorem

for $t \in (0,1)$, the solution of

$$-u''(x) = \delta_t(x), \quad u(0) = u(1) = 0$$

is $k(x,t) = \min\{x,t\} - xt$

Proof. Integrate twice, as usual:

$$u'(x) = C - \int_0^x \delta_t(s)\,ds = C - \mathbb{1}_{(t,1)}(x)$$

$$u(x) = 0 + \int_0^x C - \mathbb{1}_{(t,1)}(s)\,ds = Cx - \begin{cases} 0, & x < t \\ x - t, & x > t \end{cases}$$

But:

$$0 = u(1) = C - \begin{cases} 0, & 1 < t \\ 1 - t, & 1 > t \end{cases} = C - (1 - t)$$

so $C = 1 - t$ so:

$$u(x) = (1-t)x - \begin{cases} 0, & x < t \\ x - t, & x > t \end{cases} = \begin{cases} x(1-t), & x < t \\ t(1-x), & x > t \end{cases} = k(x,t) \qquad \square$$

- this proof is valid if we interpret $\delta_t(x)\,dx$ as a measure

physicist's solution of the BVP

Corollary

for $t \in (0,1)$, the solution of

$$-u''(x) = \phi(x), \quad u(0) = u(1) = 0$$

is $u(x) = \int_0^1 k(x,t)\phi(t)\,dt$ where $k(x,t) = \min\{x,t\} - xt$

"Proof." The map from ϕ to the solution u is linear: $u = L\phi$. Express ϕ as a linear combination of delta functions:

$$\phi(x) = \int_0^1 \phi(t)\delta_t(x)\,dt$$

Now apply L to both sides, to write the solution as a linear combination of solutions:

$$u(x) = (L\phi)(x) = \int_0^1 \phi(t) L(\delta_t(x))\,dt$$

But from previous slide, $L(\delta_t(x)) = k(x,t)$. Thus

$$u(x) = \int_0^1 \phi(t)k(x,t)\,dt = \int_0^1 k(x,t)\phi(t)\,dt \qquad \square$$

- this "proof" becomes valid in the theory of *distributions* or *generalized functions*, due to L. Schwartz

approximation of the BVP

- suppose that an integral operator $L : C([0,1]) \to C([0,1])$ has formula

$$(L\phi)(x) = \int_0^1 k(x,t)\phi(t)\,dt$$

- neither L nor k *is* a matrix, but could we approximate them by a matrix?
- one answer: finite elements
- picture of a test (hat) function $\phi(x) = \psi_k(x)$ and the output $L\psi_k$:

solution of the BVP by Fourier series

Lemma

for $n \in \mathbb{N}$, the solution of

$$-u''(x) = \sin(n\pi x), \quad u(0) = u(1) = 0$$

is $u(x) = \dfrac{1}{n^2\pi^2} \sin(n\pi x)$

- the proof is very easy?
- suppose data $\phi(x)$ is written as a Fourier sine series (Chapter 4):

$$\phi(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)$$

- then, by linearity of $(\cdot)''$, the solution of the BVP is

$$u(x) = \sum_{n=1}^{\infty} \frac{c_n}{n^2\pi^2} \sin(n\pi x)$$

solution of the BVP by Fourier series 2

- we have $\phi(x) = \sum_{n=1}^\infty c_n \sin(n\pi x)$ and $u(x) = \sum_{n=1}^\infty \frac{c_n}{n^2\pi^2} \sin(n\pi x)$
- but Fourier says: $\longleftarrow$ Chapter 4

$$c_n = 2\int_0^1 \sin(n\pi t)\phi(t)\,dt$$

- thus, assuming we may exchange the sums:

$$\begin{aligned} u(x) &= \sum_{n=1}^\infty \frac{2}{n^2\pi^2}\left(\int_0^1 \sin(n\pi t)\phi(t)\,dt\right)\sin(n\pi x) \\ &= \int_0^1 \left[\sum_{n=1}^\infty \frac{2}{n^2\pi^2}\sin(n\pi t)\sin(n\pi x)\right]\phi(t)\,dt \end{aligned}$$

- guess what! we have the kernel again:

$$k(x,t) = \min\{x,t\} - xt = \sum_{n=1}^\infty \frac{2}{n^2\pi^2}\sin(n\pi t)\sin(n\pi x)$$

- and again $u(x) = \int_0^1 k(x,t)\phi(t)\,dt$

theory in weeks 2 & 3: Chapter 2 in Saxe

to define:

- open and closed sets in metric spaces
- compact sets in metric spaces
- equicontinuity of subsets of $C([a,b])$
- separable metric spaces
- complete metric spaces
- Hilbert and Banach spaces

to prove:

- Heine-Borel theorem
- Arzela-Ascoli theorem
- $C([a,b])$ with $\|\cdot\|_\infty$ is complete (thus a Banach space)

- Assignment 2 is posted at `bueler.github.io/fa`