Poisson equation: on the disk, and in theory

calculations for week 13

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UAF Math 617 Functional Analysis

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Outline

recall: boundary-value problems for the Poisson equation

- suppose $\Omega \subset \mathbb{R}^d$ is open and has a Lipschitz, or continuously-differentiable, boundary $\partial\Omega$, with outward unit normal vector field $\mathbf{n}$

general Dirichlet problem (*strong form*)

for $f \in L^2(\Omega)$ and $g_D \in C^1(\partial\Omega)$, find $u : \Omega \to \mathbb{R}$ so that

$$\begin{aligned} -\nabla^2 u &= f \quad \text{on } \Omega \\ u &= g_D \quad \text{on } \partial\Omega \end{aligned}$$

- called *homogeneous Dirichlet* if $g_D = 0$

general Neumann problem (*strong form*)

for $f \in L^2(\Omega)$ and $g_N \in L^2(\partial\Omega)$, find $u : \Omega \to \mathbb{R}$ so that

$$\begin{aligned} -\nabla^2 u &= f \quad \text{on } \Omega \\ \nabla u \cdot \mathbf{n} &= g_N \quad \text{on } \partial\Omega \end{aligned}$$

- other notation: $\nabla u \cdot \mathbf{n} = \frac{\partial u}{\partial n} = u_n$

recall: the Sobolev spaces $H^1(\Omega)$ and $H_0^1(\Omega)$

Definition

$$H^1(\Omega) = \left\{ f \in L^2(\Omega) \,\middle|\, \begin{array}{c}\text{the weak derivative } f' \text{ exists} \\ \text{and is in } L^2(\Omega)\end{array} \right\}$$

is a Hilbert space with inner product and norm:

$$\langle f, g \rangle_{H^1} = \int_\Omega f(x) g(x)\, dx + \int_\Omega \nabla f(x) \cdot \nabla g(x)\, dx, \qquad \|f\|_{H^1} = \sqrt{\langle f, f \rangle_{H^1}}$$

Definition

$$H_0^1(\Omega) = \overline{C_c^1(\Omega)},$$

(closure is in H^1 norm, and $H_0^1(\Omega)$ is a closed subspace of $H^1(\Omega)$)

recall: the weak forms of the Poisson problems

- multiply strong form by v and use IBP to derive the weak forms:

$$-\int_\Omega (\nabla^2 u) v\, dx = -\int_{\partial\Omega} (\nabla u \cdot \mathbf{n}) v\, ds + \int_\Omega \nabla u \cdot \nabla v\, dx = \int_\Omega fv\, dx$$

weak formulation of the Poisson equation

Suppose $f \in L^2(\Omega)$ and $g_N \in L^2(\partial\Omega)$ are given. Solve one of these problems:

1. homogeneous Dirichlet: Find a solution $u \in H_0^1(\Omega)$ so that

$$\int_\Omega \nabla u \cdot \nabla v\, dx = \int_\Omega fv\, dx \quad \text{for all test functions } v \in H_0^1(\Omega)$$

2. general Neumann: Find a solution $u \in H^1(\Omega)$ so that

$$\int_\Omega \nabla u \cdot \nabla v\, dx = \int_\Omega fv\, dx + \int_{\partial\Omega} g_N v\, ds \quad \text{for all test functions } v \in H^1(\Omega)$$

recall: general Dirichlet boundary conditions

- general Dirichlet conditions have an additional complication

weak form: general Dirichlet

Suppose $f \in L^2(\Omega)$ and $g_D \in C^1(\partial\Omega)$ are given. Let $u_D \in H^1(\Omega) \cap C(\bar{\Omega})$ be any function, defined on the whole domain Ω, so that $u_D = g_D$ along $\partial\Omega$. Find a solution $u \in H_0^1(\Omega)$ so that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx - \int_\Omega \nabla u_D \cdot \nabla v \, dx$$

for all test functions $v \in H_0^1(\Omega)$.

- the actually solution to the problem is $u + u_D$
- even more generally, "mixed" conditions could partition the boundary: $\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$

Outline

1. recall: strong and weak forms of the Poisson equation
2. Poisson equation on a disk (Fourier series)
3. Poisson equation well-posedness
4. Poisson equation as minimization (and coercivity)
5. lecture content in week 13

Fourier series along the boundary of the disk

- let $D = \{x \in \mathbb{R}^2 \,:\, |x| < 1\}$ be the open unit disk, with boundary
$$\partial D = \{x \in \mathbb{R}^2 \,:\, |x| = 1\}$$
- suppose $g \in L^2(\partial D, ds)$, equivalently $g \in L^2(-\pi, \pi)$
- from Chapter 4, g has a complex Fourier series:
$$g(x) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} c_n e^{inx}, \qquad c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} g(x) e^{-inx}\, dx$$
    - regarding normalization, $\{\frac{1}{\sqrt{2\pi}} e^{inx}\}_{n \in \mathbb{Z}}$ is a complete ON sequence
- pictures of disk and g:

recall: real Fourier series

Lemma

The Fourier series $g(x) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} c_n e^{inx}$ *is a real-valued function if and only if* $c_0 \in \mathbb{R}$ *and* $c_{-n} = \overline{c_n}$.

Corollary

If $g(x)$ *is a real-valued function, and if for* $n \geq 0$ *we write* $c_n = \frac{\sqrt{2\pi}}{2}(a_n - ib_n)$, *with* $a_n, b_n \in \mathbb{R}$, *then* $g(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta)$.

Proof. Write $g(x) = \frac{1}{\sqrt{2\pi}} c_0 + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} \left(c_n e^{inx} + c_{-n} e^{-inx}\right)$. Then use the lemma, giving

$$g(x) = \frac{1}{\sqrt{2\pi}} c_0 + \frac{1}{\sqrt{2\pi}} \sum_{n=1}^{\infty} 2\,\mathrm{Re}\left(c_n e^{inx}\right)$$

Substitute $c_n = \frac{\sqrt{2\pi}}{2}(a_n - ib_n)$ for $n \geq 0$, and simplify, to get the result. □

Lemma

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos(nx)\, dx, \qquad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(x) \sin(nx)\, dx$$

harmonic functions on the disk

- on the other hand, recall that the real and imaginary parts of any complex-analytic function on the disk are harmonic
- for example, the monomials z^n, $n \geq 0$, are analytic
- note $z = re^{i\theta}$ in polar coordinates
- the real and imaginary parts of z^n are harmonic:

$$z^n = r^n e^{in\theta} = r^n(\cos(n\theta) + i\sin(n\theta))$$

Definition

for $n \geq 0$, define these harmonic functions on D:

$$\phi_n(r,\theta) = r^n\cos(n\theta), \quad \psi_n(r,\theta) = r^n\sin(n\theta)$$

harmonic functions on the disk

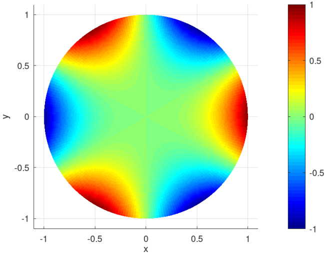
Lemma

$\phi_n(r,\theta) = r^n \cos(n\theta)$ and $\psi_n(r,\theta) = r^n \sin(n\theta)$ are harmonic.

Proof. In polar coordinates, $\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$. Then for $u = \phi_n$,

$$\nabla^2 u = n(n-1) r^{n-2} \cos(n\theta) + \frac{1}{r} n r^{n-1} \cos(n\theta) + \frac{1}{r^2}\left(-n^2 r^n \cos(n\theta)\right)$$
$$= \left[n(n-1) + n - n^2\right] r^{n-2} \cos(n\theta) = 0 r^{n-2} \cos(n\theta) = 0$$

Similarly for $u = \psi_n$. □



$\phi_3(r,\theta)$

Laplace's equation (Dirichlet's problem) on a disk

- combine these ideas to solve Laplace's equation $\nabla^2 u = 0$ on the disk
- start with Fourier series for $g \in L^2(\partial D)$; for $x \in (-\pi, \pi]$:

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

- use the coefficients to define a function on $\bar{D}$; for $0 \le r \le 1$, $\theta \in (-\pi, \pi]$:

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta) \qquad \text{for } (r,\theta) \in \bar{D}$$

Theorem

u is continuous on D, $\nabla^2 u = 0$ within D, and $\lim_{r\to 1^-} u(r,\theta) = g(\theta)$ with convergence in L^2.

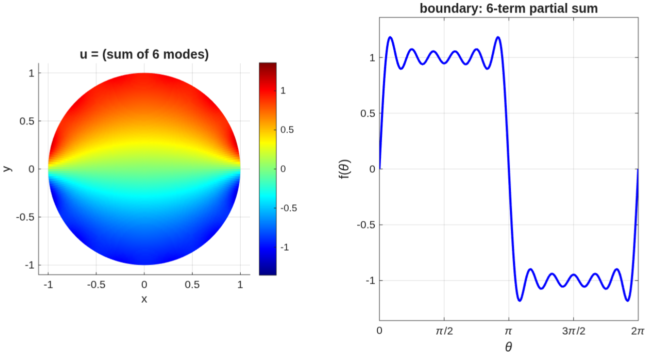
Proof. The series converges uniformly on any compact $K \subset D$, and the terms are continuous, so u is continuous on D. Compute $\|u(r,\cdot) - g(\cdot)\|_{L^2}$ and show it goes to 0 as $r \to 1$. □

Laplace's equation on a disk

- for example, suppose g is a square wave along ∂D,

$$g(x) = \begin{cases} -1, & -\pi < x < 0 \\ +1, & 0 < x < \pi \end{cases} = \sum_{j=1}^{\infty} \frac{4}{\pi(2j-1)} \sin((2j-1)\theta)$$

  - $g \in L^2(-\pi,\pi) \setminus C([-\pi,\pi])$
- to run the code which produces the movie: disklaplace.m

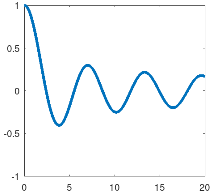


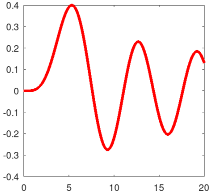
Poisson on the disk

- let's try to solve $-\nabla^2 u = f$ on the unit disk D, with $u = 0$ on ∂D
- this is the homogeneous Dirichlet problem for the Poisson equation
  - more difficult than previous series calculation; I will leave out details
- if $f \in L^2(D)$ then there is a series expansion using (co)sines and Bessel functions:

$$f(r,\theta) = \sum_{n=1}^{\infty} a_{0n} J_0(\lambda_{0n} r) + \sum_{m,n=1}^{\infty} J_m(\lambda_{mn} r)\big(a_{mn}\cos(m\theta) + b_{mn}\sin(m\theta)\big)$$

  - $J_m(r)$ is the mth Bessel function, for $m \geq 0$; below are J_0 and J_4
  - λ_{mn} is the nth positive root of $J_m(r)$
  - call this the *Fourier-Bessel expansion* of f on the disk?
  - better and general name: expansion of f in the eigenfunctions of the Laplacian





Poisson on a disk

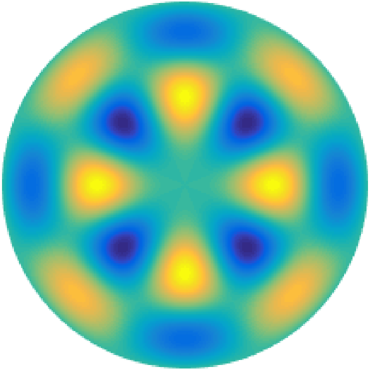
- define orthogonal functions on $L^2(D)$:

$$\phi_{mn}(r,\theta) = J_m(\lambda_{mn} r)\cos(m\theta)$$
$$\psi_{mn}(r,\theta) = J_m(\lambda_{mn} r)\sin(m\theta)$$

- these are eigenfunctions of the Laplacian:

$$\nabla^2\phi_{mn} = -\lambda_{mn}^2\phi_{mn}$$
$$\nabla^2\psi_{mn} = -\lambda_{mn}^2\psi_{mn}$$

- google "chebfun eigenfunctions of Laplacian on the disk" to see more pretty pictures



ϕ_{42}

Poisson on a disk

- now start from series expansion of source function:

$$f(r,\theta) = \sum_{n=1}^{\infty} a_{0n}\,\phi_{0n}(r,\theta) + \sum_{m,n=1}^{\infty} \left(a_{mn}\,\phi_{mn}(r,\theta) + b_{mn}\,\psi_{mn}(r,\theta)\right)$$

- solve $-\nabla^2 u = f$ using the eigenfunction property $\nabla^2 \phi_{mn} = -\lambda_{mn}^2 \phi_{mn}$

Theorem

Suppose $f \in L^2(D)$ is expanded in ϕ_{mn}, ψ_{mn} as on the last slide, with convergence in L^2. Define

$$u(r,\theta) = \sum_{n=1}^{\infty} \frac{a_{0n}}{\lambda_{0n}^2}\,\phi_{0n}(r,\theta) + \sum_{m,n=1}^{\infty} \frac{1}{\lambda_{mn}^2}\left(a_{mn}\,\phi_{mn}(r,\theta) + b_{mn}\,\psi_{mn}(r,\theta)\right).$$

Then $u \in H_0^1(D) \cap C^2(D)$ solves the both the strong form and the weak form of the Poisson equation

$$-\nabla^2 u = f$$

spectral expansions for Poisson equation

- we can combine the above techniques to solve the general Dirichlet problem for the Poisson equation on the disk:

$$-\nabla^2 u = f \;\text{ on } D, \quad u = g \;\text{ on } \partial D$$

  - *linearity* makes this possible
  - method: (i) solve $-\nabla^2 u_1 = 0$ and $u_1|_{\partial D} = g$ from Fourier series expansion of g, (ii) solve $-\nabla^2 u_2 = f$ and $u_2|_{\partial D} = 0$ from eigenfunction expansion of f, and (iii) then $u = u_1 + u_2$
- generally the above techniques are spectral expansion methods
- . . . only possible when the domain Ω has lots of symmetry, like the disk
  - also called separation of variables (for PDEs)
  - essentially the domain has to be *rectangular* in the preferred coordinates
  - for example, $D \cong [0, 1) \times (-\pi, \pi]$ in the r, θ variables

Outline

well-posedness of math problems

Definition (Hadamard, 1902)

a mathematical problem is well-posed if the solution satisfies the following properties:

1. it exists
2. it is unique
3. its value changes continuously as the data of the problem changes

Example: finite-dimensional linear systems

Suppose $A \in \mathbb{C}^{m\times m}$ is invertible and $\mathbf{b} \in \mathbb{C}^m$. The problem of solving $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x} \in \mathbb{C}^m$ is well posed because

1. exists: $\mathbf{x} = A^{-1}\mathbf{b}$
2. unique: if $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_2 = \mathbf{b}$ then $A(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0}$ so $\mathbf{x}_1 = \mathbf{x}_2$
3. continuous in the data $\mathbf{b}$: if $A\mathbf{x}_1 = \mathbf{b}_1$ and $A\mathbf{x}_2 = \mathbf{b}_2$ then
$$\|\mathbf{x}_1 - \mathbf{x}_2\| = \|A^{-1}\mathbf{b}_1 - A^{-1}\mathbf{b}_2\| \le \|A^{-1}\|\|\mathbf{b}_1 - \mathbf{b}_2\|$$

well-posedness of math problems

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1. exists: $\mathbf{x} = A^{-1}\mathbf{b}$
2. unique: if $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_2 = \mathbf{b}$ then $A(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0}$ so $\mathbf{x}_1 = \mathbf{x}_2$
3. continuous in the data $\mathbf{b}$: if $A\mathbf{x}_1 = \mathbf{b}_1$ and $A\mathbf{x}_2 = \mathbf{b}_2$ then
   $$\|\mathbf{x}_1 - \mathbf{x}_2\| = \|A^{-1}\mathbf{b}_1 - A^{-1}\mathbf{b}_2\| \le \|A^{-1}\|\|\mathbf{b}_1 - \mathbf{b}_2\|$$

are Poisson problems well-posed?

- can we prove that a boundary-value problem for Poisson is well-posed?
- solving it in particular cases, e.g. for the disk, will not help generally
- recall: two important theorems which will help

Theorem (Poincaré-Friedrichs inequality)

Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with Lipschitz boundary, or an open set which is bounded in one direction. There is $C > 0$, depending on Ω, so that

$$\|f\|_{L^2} \le C\|\nabla f\|_{L^2} \quad \text{for all } f \in H_0^1(\Omega).$$

Theorem (Riesz-Fréchet representation theorem)

If $(V, \langle\cdot,\cdot\rangle)$ is a (real) Hilbert space, and if $\ell \in V'$ is a bounded linear functional, then there exists $u \in V$ so that

$$\langle u, v\rangle = \ell(v) \quad \text{for all } v \in V$$

Riesz representation route to well-posedness

Theorem

The weak-form homogeneous Dirichlet problem, for $u \in H_0^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \text{for all } v \in H_0^1(\Omega),$$

is well-posed.

Proof. Let $\langle\langle u, v\rangle\rangle = \int_\Omega \nabla u \cdot \nabla v \, dx$. We show that this is also an inner product on $H_0^1(\Omega)$.[1] It is clearly nonnegative, symmetric, and bilinear, and furthermore it is nondegenerate: $\langle\langle u, u\rangle\rangle = 0$ implies

$$\|u\|_{H^1}^2 = \int_\Omega u^2 \, dx + \int_\Omega |\nabla u|^2 \, dx \le (C^2 + 1) \int_\Omega |\nabla u|^2 \, dx = (C^2 + 1)\langle\langle u, u\rangle\rangle = 0,$$

by the Poincaré-Friedrichs inequality, so $u = 0$. This also shows that $\langle\langle \cdot, \cdot\rangle\rangle$ and $\langle \cdot, \cdot\rangle$ give equivalent norms. Thus $(H_0^1(\Omega), \langle\langle \cdot, \cdot\rangle\rangle)$ is a real Hilbert space.

[1] The usual inner product is $\langle u, v\rangle = \int_\Omega uv \, dx + \int_\Omega \nabla u \cdot \nabla v \, dx$.

proof of Riesz representation route 2

Proof, continued. On the other hand, the right-hand side of the weak form defines a linear functional, $\ell : H_0^1(\Omega) \to \mathbb{R}$, $\ell(v) = \int_\Omega fv\,dx$. This is bounded by Cauchy-Schwarz,

$$|\ell(v)| \leq \int_\Omega |fv|\,dx \leq \|f\|_{L^2}\|v\|_{L^2} \leq \|f\|_{L^2}\|v\|_{H^1},$$

so $\ell \in H_0^1(\Omega)'$. By the Riesz-Fréchet representation theorem there is a unique $u \in H_0^1(\Omega)$ so that

$$\langle\langle u, v\rangle\rangle = \ell(v).$$

This exactly says that a solution to the weak form problem exists, and that is it unique.

Finally, to show that the solution is continuous with respect to the data f, observe that the solution process is a well-defined linear map. That is, there is a linear map $T : L^2(\Omega) \to H_0^1(\Omega)$ defined by[2]

$$Tf = u$$

Recall that a linear map is continuous if and only if it is bounded. We will show that

$$\|u\|_{H^1} = \|Tf\|_{H^1} \leq c\|f\|_{L^2}.$$

[2] Conceptually, $T = (-\nabla^2)^{-1}$. But boundary conditions are involved!

proof of Riesz representation route 3

Proof, continued. To show that T is bounded we substitute u for v in the weak form, use a previous calculation, and apply Cauchy-Schwarz:

$$\begin{aligned}\|u\|_{H^1}^2 &\le (C^2+1)\langle\langle u,u\rangle\rangle = (C^2+1)\int_\Omega \nabla u\cdot\nabla u\,dx \\ &= (C^2+1)\int_\Omega fu\,dx \le (C^2+1)\|f\|_{L^2}\|u\|_{L^2} \le (C^2+1)\|f\|_{L^2}\|u\|_{H^1}\end{aligned}$$

Now divide by the norm of u to get a bound:

$$\|u\|_{H^1} \le (C^2+1)\|f\|_{L^2}.$$

This shows that the solution map $Tf = u$ is continuous, and concludes the proof of well-posedness. □

- a bound like $\|u\|_{H^1} \le c\|f\|_{L^2}$ is often called an *a priori* bound
- an *a priori* bound is sufficient to conclude continuity (w.r.t. f) for *linear* problems

the Neumann problem is *not* well-posed

- recall: the general weak-form Neumann problem is to find $u \in H^1(\Omega)$ s.t.

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx + \int_{\partial\Omega} g_N v \, ds \qquad \text{for all } v \in H^1(\Omega)$$

  - strong form: $-\nabla^2 u = f$ on Ω, $\nabla u \cdot \mathbf{n} = g_N$ on $\partial\Omega$
- this has a uniqueness failure!

Lemma

If this problem has a solution then it has infinitely-many solutions.

Proof. Let $u \in H^1(\Omega)$ be a solution and consider any $C \in \mathbb{R}$. Then $\tilde{u}(x) = u(x) + C$ is also in $H^1(\Omega)$, with the same gradient, and thus it satisfies the same weak form. □

well-posed modifications of the Neumann problem

- there are two well-known ways to fix this uniqueness issue
- proofs for this theorem are skipped, but
  - note the strong form for ❶: $\boxed{-\nabla^2 u + \alpha u = f}$ on Ω, $\nabla u \cdot \mathbf{n} = g_N$ on $\partial\Omega$
  - on Assignment 8: prove ❶ via the Riesz-Fréchet representation theorem

Theorem

Suppose $\Omega \subset \mathbb{R}^2$ has a Lipschitz or C^1 boundary, $f \in L^2(\Omega)$, and $g_N \in L^2(\partial D)$. The following weak-form Neumann-type problems for $u \in H^1(\Omega)$ are well-posed:

1. *Neumann problem for the Helmholtz equation: Suppose $\alpha > 0$.*

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega \alpha u v \, dx = \int_\Omega f v \, dx + \int_{\partial\Omega} g_N v \, ds \quad \text{for all } v \in H^1(\Omega)$$

2. *Neumann problem with auxiliary condition:*

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx + \int_{\partial\Omega} g_N v \, ds \quad \text{for all } v \in H^1(\Omega)$$

and so that $\int_\Omega u \, dx = 0$, equivalently $\langle u, c\rangle = 0$ for all constant functions c.

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minimization viewpoint

- the weak form of the Poisson equation can be regarded as the critical-point condition, "$J'(u) = 0$", for the problem of minimizing a real-valued function:

$$\min_{x \in S} J(u)$$

- however, we are working in ∞-dimensions, so optimization requires functional analysis
- for this material: D. Braess (2007), *Finite Elements*, 3rd ed., Cambridge U. Press

picture:

characterization theorem: minimization $\iff$ weak form

- this idea is clearest in an abstract formulation

Theorem (characterization theorem)

Let V be a real vector space. Suppose

$$a : V \times V \to \mathbb{R}$$

is a symmetric and nonnegative ($a(v,v) \geq 0$) bilinear function. Suppose $\ell : V \to \mathbb{R}$ is a linear functional. Define the quadratic objective function

$$J(v) = \frac{1}{2} a(v,v) - \ell(v).$$

Then $u \in V$ is a minimizer of J if and only if

$$a(u,v) = \ell(v) \quad \text{for all } v \in V.$$

- this theorem does *not* assert existence or uniqueness; it *characterizes u*
- this theorem does *not* require V to be complete
- this theorem does *not* require the bilinear form a to be positive definite

proof of the characterization theorem 1

Proof. Fix $u, v \in V$ and suppose $t \in \mathbb{R}$. Define

$$\begin{aligned} g(t) = J(u+tv) &= \frac{1}{2} a(u+tv, u+tv) - \ell(u+tv) \\ &= \frac{1}{2}\Big(a(u,u) + 2t\, a(u,v) + t^2 a(v,v)\Big) - \ell(u) - t\ell(v) \\ &= J(u) + t\Big[a(u,v) - \ell(v)\Big] + \frac{1}{2} t^2 a(v,v). \end{aligned}$$

Now if u solves the weak form ($a(u,v) - \ell(v) = 0$) then, using $t = 1$ and nonnegativity of a,

$$J(u+v) = g(1) = J(u) + \frac{1}{2} a(v,v) \geq J(u).$$

This holds for any $v \in V$, so u is a minimizer of J.

Proof, cont.

Conversely, suppose J is minimized by u. By assumption, the real function $g(t) = J(u + tv)$ has a minimum at $t = 0$. However, we have written g as a quadratic polynomial in t, with derivative

$$g'(t) = a(u, v) - \ell(v) + t\, a(v, v).$$

Since u is a minimizer, $g'(0) = 0$, which shows that $a(u, v) - \ell(v) = 0$, so u satisfies the weak form. □

- example: the homogeneous Dirichlet problem $-\nabla^2 u = f$, $u\big|_{\partial\Omega} = 0$
- here: $V = H_0^1(\Omega)$, $a(u,v) = \int_\Omega \nabla u \cdot \nabla v\, dx$, $\ell(v) = \int_\Omega fv\, dx$

Corollary

The element $u \in H_0^1(\Omega)$ solves the weak-form problem

$$\int_\Omega \nabla u \cdot \nabla v\, dx = \int_\Omega fv\, dx \quad \textit{for all } v \in H_0^1(\Omega),$$

if and only if it is a minimizer of the energy functional $J : H_0^1(\Omega) \to \mathbb{R}$:

$$u \quad \overset{\min}{\leftarrow} \quad J(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\, dx - \int_\Omega fv\, dx.$$

- again, we are *not* (here) asserting existence of u
- in the loaded elastic membrane application of the Poisson equation, $J(v)$ is the sum of the elastic ($\frac{1}{2}\int_\Omega |\nabla v|^2\, dx$) and the gravitational/load ($-\int_\Omega fv\, dx$) potential energy

minimization route to well-posedness

- on a Hilbert space, if the bilinear form satisfies certain inequalities, then the minimizer exists and is unique

Theorem (existence of a minimizer)

Let H be a Hilbert space. Suppose $a : H \times H \to \mathbb{R}$ is a symmetric bilinear function. Assume it is coercive in the sense that there is $\alpha > 0$ so that

$$a(v, v) \geq \alpha \|v\|^2 \quad \forall v \in H.$$

Assume it is continuous in the sense that there is $C \geq 0$ so that

$$|a(u, v)| \leq C\|u\|\|v\| \quad \forall v \in H.$$

Suppose $\ell \in H'$. Then there exists $u \in H$ which is the unique minimizer of J:

$$u \quad \overset{\min}{\leftarrow} \quad J(v) = \frac{1}{2}a(v, v) - \ell(v).$$

- coercivity is also called (uniform) *ellipticity*
- Braess (2007) *Finite Elements*, calls this is a Lax-Milgram theorem. I am not sure if that is right or wrong historically, but the usual Lax-Milgram theorem is *not* about minimization.

Proof. By coercivity and the definition of the H' norm,

$$\begin{aligned} J(v) &= \frac{1}{2}a(v,v) - \ell(v) \geq \frac{\alpha}{2}\|v\|_H^2 - \|\ell\|_{H'}\|v\|_H \\ &= \frac{1}{2\alpha}\left(\alpha\|v\|_H - \|\ell\|_{H'}\right)^2 - \frac{\|\ell\|_{H'}^2}{2\alpha} \\ &\geq -\frac{\|\ell\|_{H'}^2}{2\alpha}. \end{aligned}$$

This shows J is bounded below. Thus

$$\mu = \inf\{J(v) \,:\, v \in V\}$$

exists as a real number.

We will show that there is a $u \in V$ so that $J(u) = \mu$, namely that a minimizer achieves the infimum.

proof of the minimizer-existence theorem 2

Proof, cont.

Suppose (v_n) is a sequence in V such that $J(v_n) \to \mu$, that is, it is a minimizing sequence. Then[3]

$$\begin{aligned}
\alpha \|v_n - v_m\|_H^2 &\le a(v_n - v_m, v_n - v_m) \\
&= 2a(v_n, v_n) + 2a(v_m, v_m) - a(v_n + v_m, v_n + v_m) \\
&= 2a(v_n, v_n) - 4\ell(v_n) + 2a(v_m, v_m) - 4\ell(v_n) \\
&\qquad - 4a\left(\frac{v_n + v_m}{2}, \frac{v_n + v_m}{2}\right) + 8\ell\left(\frac{v_n + v_m}{2}\right) \\
&= 4J(v_n) + 4J(v_m) - 8J\left(\frac{v_n + v_m}{2}\right) \\
&\le 4J(v_n) + 4J(v_m) - 8\mu.
\end{aligned}$$

But since $J(v_n) \searrow \mu$ and $J(v_m) \searrow \mu$, this inequality shows $\|v_n - v_m\| \to 0$, so (v_n) is Cauchy.

[3]We used $J\left(\frac{v_n+v_m}{2}\right) \ge \mu$ in the last step. This argument generalizes to convex $C \subset H$.

proof of the minimizer-existence theorem 3

Proof cont.

Since H is complete,[4] $u = \lim_{n\to\infty} v_n \in H$ exists. The continuity of J at u, which follows from the continuity of a and ℓ, now shows

$$J(u) = J\left(\lim_{n\to\infty} v_n\right) = \lim_{n\to\infty} J(v_n) = \mu,$$

so a minimizer exists.

Now suppose that u_1, u_2 are both minimizers, so $J(u_i) = \mu$. The previous calculation[5] applied to u_1 and u_2 gives zero exactly:

$$\alpha\|u_1 - u_2\|^2 \le 4J(u_1) + 4J(u_2) - 8\mu = 0.$$

Thus $u_1 = u_2$, so the solution is unique. □

[4] Here the argument would need $C \subset H$ to be closed, in addition to convex.

[5] Here we are seeing that J is *strictly* convex, thus the minimizer is unique.

well-posedness for the Poisson problem

- same example: homogeneous Dirichlet problem

Corollary

The solution $u \in H_0^1(\Omega)$ of the weak-form problem

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \text{for all } v \in H_0^1(\Omega)$$

exists and is unique.

Proof. Apply the characterization theorem to turn the weak form into a minimization. Show coercivity and continuity of $a(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx$. (Coercivity will need the Poincar'e-Friedrichs inequality.) Then apply the last theorem to show the minimizer exists and is unique. □

- this strategy works nearly the same for the Neumann problem for the Helmholtz equation (earlier)

understanding minimization in ∞-dimensions

- minimization in ∞-dimensions is a big story, which we have only started
- historically called the *direct approach* to the calculus of variations
- for well-posedness, it is an alternative to a Riesz-Fréchet representation theorem argument
    - a separate argument is needed to complete the definition of well-posedness, i.e. continuity of u with respect to changes in the data f
- this can be generalized to Banach spaces (not shown)
- this can be generalized to minimization over closed convex sets (not shown, but see hints)
    - and even further to variational inequalities
- it is an especially good approach for elliptic PDEs, but it also works for other problems, including some time-dependent problems, under the name Euler-Lagrange equations for action functionals

Outline

1. recall: strong and weak forms of the Poisson equation
2. Poisson equation on a disk (Fourier series)
3. Poisson equation well-posedness
4. Poisson equation as minimization (and coercivity)
5. lecture content in week 13

lecture content in week 13

- this material is only from slides; see also the week 12 slides

to know from these slides:

- solution to Dirichlet problem for Laplace's equation on the disk by Fourier series ($-\nabla^2 u = 0$ on D, $u|_{\partial D} = g$)
- solution to homogeneous Dirichlet problem for Poisson equation on the disk by eigenfunction expansion ($-\nabla^2 u = f$ on D, $u|_{\partial D} = 0$)
- Hadamard's definition of well-posed problem
- example of well-posedness: invertible finite-dimensional linear systems ($A\mathbf{x} = \mathbf{b}$)
- Riesz representation theorem proof of well-posedness of homogeneous Dirichlet problem for Poisson equation on any domain
- why the Neumann problem *fails uniqueness*, and that modified problems are well-posed
- characterization theorem relating an abstract quadratic optimization problem to the corresponding weak form
- existence and uniqueness for $J(v) = \frac{1}{2}a(v,v) - \ell(v)$ if the bilinear form is coercive and continuous