Poisson equation and Sobolev spaces

calculations for week 12 (*version 2*)

Ed Bueler

UAF Math 617 Functional Analysis

Spring 2026

Outline

1. the Poisson equation: $-\nabla^2 u = f$
2. the Poisson equation in weak form: $\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx$
3. a quick sketch of the finite element method
4. defining the Sobolev spaces: $H^k(\Omega)$
5. more about $H^1(\Omega)$
6. lecture content in week 12

the Poisson equation

- let $\Omega \subset \mathbb{R}^d$ be open; denote points by $x = (x_1, \dots, x_d) \in \Omega$
  - assume $d \in \{1, 2, 3\}$ ... I will draw pictures for $d = 2$
- assume the boundary of Ω is not too crazy (more below)
- assume $f \in L^2(\Omega)$, with real values, is (square-)integrable

Poisson equation

find $u = u(x)$ so that

$$-\nabla^2 u = f \qquad \text{on } \Omega$$

- next slide defines "∇^2" ...
- picture:

the Laplacian

- given f, finding u from the Poisson equation $-\nabla^2 u = f$ is a problem of inverting the operator $-\nabla^2$

Definition (Laplacian operator)

for $\Omega \subset \mathbb{R}^d$ open and $u : \Omega \to \mathbb{R}$, the *Laplacian* of u is

$$\nabla^2 u = \nabla \cdot (\nabla u) = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_d^2}$$

- in each dimension:

$$\begin{aligned} d = 1 : &\qquad \nabla^2 u = u'' \\ d = 2 : &\qquad \nabla^2 u = u_{xx} + u_{yy} \\ d = 3 : &\qquad \nabla^2 u = u_{xx} + u_{yy} + u_{zz} \end{aligned}$$

tool: divergence theorem

- $\nabla^2 = \nabla \cdot \nabla$ usually appears in models because
  1. a flux is proportional to a gradient, and
  2. a divergence connects the interior behavior to a boundary integral of the flux
- ...examples to come
- we will need the divergence theorem from the weeks 10&11 slides:

$$\int_\Omega \nabla \cdot \mathbf{V}\, dm = \int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n}\, ds,$$

  - recall that this theorem applies to $\Omega \subset \mathbb{R}^d$ if $\partial\Omega$ is Lipschitz or C^1
- when using this, $\mathbf{V}$ will usually be the gradient of a scalar function
- recall the boundary integral $\int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n}\, ds$ is the total flux of $\mathbf{V}$ outward through the boundary

Laplace's equation, and harmonic functions

- in the absence of any boundary conditions there is an infinite-dimensional space of solutions to $-\nabla^2 u = f$ on Ω

Definition (Laplace's equation = potential equation)

$$\nabla^2 u = 0$$

a solution of this $f = 0$ equation is called *harmonic*

- the set of harmonic functions on Ω is an ∞-dimensional vector space of smooth functions:

$$\mathcal{H}(\Omega) = \left\{ w \,:\, \nabla^2 w = 0 \right\} \subset C^\infty(\Omega)$$

- complex analysis is, essentially, the study of this space

boundary-value problems for the Poisson equation

- however, we want to build models which make *one* prediction
- the Poisson equation can be used to model heat conduction, electrostatic potential, elastic membranes, equilibrium distributions for random walks, or many other physical phenomena
- we will focus on *boundary-value problems* that pick-out one solution u

Dirichlet problem for the Poisson equation

for $f \in L^2(\Omega)$ and $g_D \in C^1(\partial\Omega)$, find $u : \Omega \to \mathbb{R}$ so that

$$\begin{aligned} -\nabla^2 u &= f \quad \text{on } \Omega \\ u &= g_D \quad \text{on } \partial\Omega \end{aligned}$$

boundary-value problems for the Poisson equation

- instead of u itself, the gradient ∇u could be set to known values along the boundary
- in models, the gradient is often proportional to a physical flux

Neumann problem for the Poisson equation

for $f \in L^2(\Omega)$ and $g_N \in L^2(\partial\Omega)$, find $u : \Omega \to \mathbb{R}$ so that

$$\begin{aligned} -\nabla^2 u &= f \quad \text{on } \Omega \\ \nabla u \cdot \mathbf{n} &= g_N \quad \text{on } \partial\Omega \end{aligned}$$

where $\mathbf{n}$ is the outward unit normal vector field along $\partial\Omega$

- other notation for the (scalar) directional derivative:

$$\nabla u \cdot \mathbf{n} = \frac{\partial u}{\partial n} = u_n$$

- one could have Dirichlet conditions on part of the boundary and Neumann on another part, as a disjoint union:

$$\partial\Omega = \partial_D\Omega \cup \partial_N\Omega$$

application: temperature of a conductive solid

- Fourier's law for heat conduction in solids says that the heat flux is

$$\mathbf{q} = -k\nabla u,$$

  where u is the temperature, $\mathbf{q}$ is the heat flux, and $k > 0$ is the conductivity
- consider a solid with a heat capacity $c > 0$ (heat energy gained by increase in temperature) and a mass density $\rho > 0$ (mass per unit volume)
- given a heat source f within the domain Ω, conservation of energy says

$$c\rho \frac{\partial u}{\partial t} = -\nabla \cdot \mathbf{q} + f \quad \iff \quad \frac{d}{dt}\left(\int_\Omega c\rho\, u\, dx\right) = -\int_{\partial\Omega} \mathbf{q}\cdot\mathbf{n}\, ds + \int_\Omega f\, dx$$

- assume steady state, so $\partial/\partial t = d/dt = 0$
- if k is constant then $-\nabla\cdot\mathbf{q} = -\nabla\cdot(-k\nabla u) = k\nabla^2 u$
- we derive Poisson's equation, the (steady) *heat* or *diffusion equation*

$$0 = k\nabla^2 u + f \qquad \text{within } \Omega$$

- boundary conditions are needed! (coming soon)

application: deflection of an elastic membrane

- suppose $z = u(x, y)$ is a vertical displacement of an elastic membrane over a domain $\Omega \subset \mathbb{R}^2$
- suppose that along $\partial\Omega$ we have put a rigid frame with height $z = g(x, y)$
- suppose that the slopes along the graph of g are small
- suppose that the membrane is loaded by a force f (per unit area)
- group the elastic properties of the membrane into a constant $\kappa > 0$
- then the vertical displacement satisfies Poisson's equation

$$-\kappa \nabla^2 u = f \qquad \text{within } \Omega$$

  subject to $u = g$ along $\partial\Omega$
- the standard derivation minimizes the total elastic energy of the membrane,
- the Poisson equation is saying that the derivative is zero at the minimizer

applications: Poisson equation boundary value problems

pictures for $d = 1$ steady heat in a rod, with insulated ends:

pictures for $d = 2$ membrane over a frame:

Outline

1. the Poisson equation: $-\nabla^2 u = f$
2. the Poisson equation in weak form: $\displaystyle\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega fv \, dx$
3. a quick sketch of the finite element method
4. defining the Sobolev spaces: $H^k(\Omega)$
5. more about $H^1(\Omega)$
6. lecture content in week 12

deriving the weak form of the Poisson equation

- first I'll describe *how* to get the weak formulation, then I'll explain *why*
- let's start from the Poisson equation, $-\nabla^2 u = f$
- we seek the solution function $u : \Omega \to \mathbb{R}$
- remember that $\nabla^2 = \nabla \cdot \nabla$, so the problem is:

$$-\nabla \cdot (\nabla u) = f \qquad \text{within } \Omega$$

- suppose $v : \Omega \to \mathbb{R}$ is another function, which we call a *test function*
- multiply the Poisson equation by v and integrate over Ω:

$$-\int_\Omega \nabla \cdot (\nabla u) v \, dx \overset{*}{=} \int_\Omega f v \, dx$$

- apply the integration by parts rule from the weeks 10&11 slides:

$$\int_\Omega \phi \nabla \cdot \mathbf{V} \, dx = \int_{\partial\Omega} \phi \mathbf{V} \cdot \mathbf{n} \, ds - \int_\Omega \nabla\phi \cdot \mathbf{V} \, dx$$

  - we'll apply this on the left side of $*$, with $\phi = v$ and $\mathbf{V} = \nabla u$

deriving the weak form of the Poisson equation

- we get:

$$-\int_{\partial\Omega} v\,(\nabla u)\cdot \mathbf{n}\,ds + \int_\Omega \nabla v\cdot\nabla u\,dx = \int_\Omega fv\,dx$$

- rearrange to get:

$$\int_\Omega \nabla u\cdot\nabla v\,dx = \int_\Omega fv\,dx + \int_{\partial\Omega} v\,(\nabla u\cdot\mathbf{n})\,ds$$

  - the integral $\int_\Omega fv\,dx$ is *data*, because f is given
  - existence and uniqueness of the solution u, essentially requires turning the boundary integral into data also
- there are two options for removing u from the boundary integral:
  1. for Dirichlet boundary conditions, only use test functions v which have zero value along $\partial\Omega$
  2. for Neumann boundary conditions, replace $\nabla u\cdot\mathbf{n}$ with the data g_N

ALMOST the weak form of the Poisson equation

weak formulation of the Poisson equation

Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with Lipschitz boundary, and suppose $f \in L^2(\Omega)$ is given. Solve one of these problems:

1. Find a solution $u \in C_c^1(\Omega)$ so that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \text{(homogeneous Dirichlet)}$$

   for all test functions $v \in C_c^1(\Omega)$.

2. Find a solution $u \in C^1(\Omega)$ so that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx + \int_{\partial\Omega} g_N v \, ds \qquad \text{(general Neumann)}$$

   for all test functions $v \in C^1(\Omega)$.

- this is not the right way to do it!
- asking for $C^1(\Omega)$ is too strong!

weak formulation of the Poisson equation

Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with Lipschitz boundary, and suppose $f \in L^2(\Omega)$ is given. Solve one of these problems:

1. Find a solution $u \in H_0^1(\Omega)$ so that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \text{(homogeneous Dirichlet)}$$

   for all test functions $v \in H_0^1(\Omega)$.

2. For $g_N \in L^2(\partial\Omega)$, find a solution $u \in H^1(\Omega)$ so that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx + \int_{\partial\Omega} g_N v \, ds \qquad \text{(general Neumann)}$$

   for all test functions $v \in H^1(\Omega)$.

- I will define the *Sobolev space* $H^1(\Omega)$ in the next section
- for general Dirichlet conditions there is an additional complication

what is the “weak formulation”?

observation 1.

The strong form is pointwise: $(-\nabla^2 u)(x) = f(x)$ holds at every $x \in \Omega$. By contrast, the weak form is about averages using test functions. That is, the weak form holds on the supports of the v functions.

- here’s a cartoon of where the equation is enforced:

observation 2.

The strong form refers to the second derivatives, the Laplacian, of the unknown solution u. By contrast, only the first derivatives of u, namely the gradient, is needed in the weak form.

general Dirichlet boundary conditions

- Suppose $\Omega \subset \mathbb{R}^d$ is an open, bounded domain with Lipschitz boundary.
- Suppose $f \in L^2(\Omega)$ and $g_D \in L^2(\partial\Omega)$ are given.
- Recall the strong form of the general Dirichlet problem:

$$-\nabla^2 u = f \quad \text{on } \Omega, \qquad \text{and } u = g_D \quad \text{on } \partial\Omega$$

- For this weak form we substitute $u = \tilde{u} + u_D$ into the homogeneous form.

weak form general Dirichlet problem

Let $u_D \in H^1(\Omega) \cap C(\bar{\Omega})$ be any function, *defined on the whole domain* Ω, so that $u_D = g_D$ along $\partial\Omega$. Find a solution $\tilde{u} \in H_0^1(\Omega)$ so that

$$\int_\Omega \nabla\tilde{u} \cdot \nabla v\, dx = \int_\Omega fv\, dx - \int_\Omega \nabla u_D \cdot \nabla v\, dx$$

for all test functions $v \in H_0^1(\Omega)$.

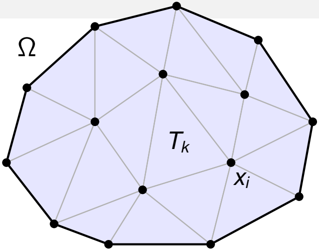
- The new right-hand side is data. The solution is $u = \tilde{u} + u_D \in H^1(\Omega)$.

Outline

1. the Poisson equation: $-\nabla^2 u = f$
2. the Poisson equation in weak form: $\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx$
3. a quick sketch of the finite element method
4. defining the Sobolev spaces: $H^k(\Omega)$
5. more about $H^1(\Omega)$
6. lecture content in week 12

what is a finite element method?

- suppose $\Omega \subset \mathbb{R}^2$ is a polygon
- put a mesh $\mathcal{T}$ of triangles on Ω
- denote nodes x_i and triangles T_k



- we call $f(x,y) = a + bx + cy$ a *linear polynomial* in two variables

Definition (the continuous P_1 finite element space over $\mathcal{T}$)

$$P_1 = \left\{ f \in C(\Omega) \,:\, f\big|_{T_k} \text{ is a linear polynomial} \right\}$$

- the approximate solution u_h, and the test functions v_h, will be from the following finite-dimensional subspace of V:

$$V_h = \left\{ f \in P_1 \,:\, f\big|_{\partial\Omega} = 0 \right\} \subset V$$

- $w_h \in V_h$ is determined by finitely-many numbers, the nodal values

what is a finite element method (FEM)?

weak form of the Poisson equation (homogeneous Dirichlet)

Find a solution $u \in V = H_0^1(\Omega)$ so that

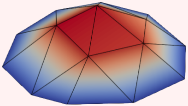
$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \qquad \forall v \in V$$

- FEM solves the same weak form, but over the finite-dim'l subspace V_h

a FEM for the Poisson equation (homogeneous Dirichlet)

Find a solution $u_h \in V_h = \left\{ f \in P_1 \,:\, f\big|_{\partial\Omega} = 0 \right\}$ so that

$$\int_\Omega \nabla u_h \cdot \nabla v_h \, dx = \int_\Omega f v_h \, dx \qquad \forall v_h \in V_h$$

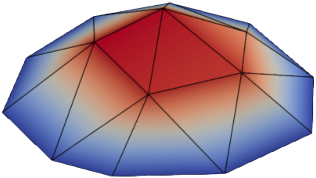


- a finite-dimensional vector $\mathbf{u} \in \mathbb{R}^N$ holds values of u_h at nodes x_i
- a computer program loops over the triangles T_k, computing the above integrals, to assemble a linear system in $\mathbb{R}^N$: $\quad A\mathbf{u} = \mathbf{b}$

finite element method automation

here is an FEM workflow:

- define a mesh in Gmsh format (.msh)
- solve the finite element problem using a Python program (.py) which calls the Firedrake library
- visualize the result using Paraview (.pvd)



blob.msh:

```
$MeshFormat
4.1 0 8
$EndMeshFormat
$PhysicalNames
2
1 1 "Boundary"
2 2 "Domain"
$EndPhysicalNames
$Entities
0 1 1 0
1 -0.5 0.0 0.0 4.0 3.5 0.0 1 1 0
1 -0.5 0.0 0.0 4.0 3.5 0.0 1 2 1 1
...
```

blob.py:

```python
from firedrake import *
mesh = Mesh("blob.msh")
Vh = FunctionSpace(mesh, "P", 1)
uh = Function(Vh, name="u_h")
vh = TestFunction(Vh)
f = Constant(1.0)
F = dot(grad(uh), grad(vh)) * dx - f * vh * dx
zero = DirichletBC(Vh, Constant(0.0), (1,))
solve(F == 0, uh, bcs=[zero,])
VTKFile("result.pvd").write(uh)
```

see these codes at github.com/bueler/fa/tree/main/assets/slides/S26/codes

Outline

1. the Poisson equation: $-\nabla^2 u = f$
2. the Poisson equation in weak form: $\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega fv \, dx$
3. a quick sketch of the finite element method
4. defining the Sobolev spaces: $H^k(\Omega)$
5. more about $H^1(\Omega)$
6. lecture content in week 12

recall weak derivatives

- given a bounded open set $\Omega \subset \mathbb{R}^d$
- recall the definition: $L^2(\Omega) = \{f : \Omega \to \mathbb{R} \,\big|\, \int_\Omega |f|^2 < \infty\}$
  - if Ω is bounded then $L^2(\Omega) \subset L^1(\Omega)$
- recall our multi-index derivative notation "$D^\alpha \varphi$", from the weeks 10&11 slides, and this major definition:

Definition (weak derivative)

if $f \in L^1_{\text{loc}}(\Omega)$, and if $g \in L^1_{\text{loc}}(\Omega)$ satisfies

$$\int_\Omega g\,\varphi\,dm = (-1)^{|\alpha|} \int_\Omega f\,D^\alpha \varphi\,dm \qquad \text{for all } \varphi \in C_c^\infty(\Omega)$$

then we say g is the α*th weak derivative* of f, and we write $D^\alpha f = g$ a.e.

- *example:* f has a weak derivative $g = \partial f/\partial x_k \in L^1(\Omega)$ if

$$\int_\Omega g\,\varphi\,dm = -\int_\Omega f\,\frac{\partial \varphi}{\partial x_k}\,dm \qquad \text{for all } \varphi \in C_c^\infty(\Omega)$$

and $\int_\Omega |g|\,dm < \infty$

Definition (Sobolev space $H^1(\Omega)$)

$$H^1(\Omega) = \left\{ f \in L^2(\Omega) \,\middle|\, \begin{matrix} \text{the weak partial derivatives } \frac{\partial f}{\partial x_k} \text{ exist} \\ \text{and are in } L^2(\Omega) \text{ for } k = 1, \dots, d \end{matrix} \right\}$$

is a Hilbert space with inner product:

$$\langle f, g \rangle_{H^1} = \int_\Omega f(x) g(x)\, dx + \int_\Omega \nabla f(x) \cdot \nabla g(x)\, dx$$

and norm: $\|f\|_{H^1} = \sqrt{\langle f, f \rangle_{H^1}} = \left(\int_\Omega |f|^2 + \int_\Omega |\nabla f|^2 \right)^{1/2}$

- *example on $\mathbb{R}^2$:* Consider the unit square $\Omega = (0,1)^2$ and the function $f(x) = x_1 + x_2$. It has gradient $\nabla f = \langle 1, 1 \rangle$, and norm $\|f\|_{H^1} = \sqrt{19/6}$:

$$\begin{aligned}
\|f\|_{H^1}^2 &= \int_\Omega |x_1 + x_2|^2\, dx + \int_\Omega |\langle 1, 1 \rangle|^2\, dx \\
&= \int_0^1 \int_0^1 x_1^2 + 2x_1 x_2 + x_2^2 \, dx_1\, dx_2 + \int_0^1 \int_0^1 2\, dx_1\, dx_2 = \frac{7}{6} + 2 = \frac{19}{6}
\end{aligned}$$

more Sobolev spaces

Definition (Sobolev space $H^k(\Omega)$)

$$H^k(\Omega) = \left\{ f \in L^2(\Omega) \,\middle|\, \begin{array}{l}\text{the weak derivatives } D^\alpha f \text{ exist} \\ \text{and are in } L^2(\Omega) \text{ for all } |\alpha| \le k\end{array} \right\}$$

is a Hilbert space with inner product:

$$\langle f, g \rangle_{H^k} = \sum_{|\alpha| \le k} \int_\Omega (D^\alpha f)(x)(D^\alpha g)(x)\,dx$$

and norm: $\|f\|_{H^k} = \sqrt{\langle f, f \rangle_{H^k}}$

- "H" is for Hilbert
- we may write $H^0(\Omega) = L^2(\Omega)$, because $D^\alpha f = f$ if $\alpha = (0, \dots, 0)$
- some sources use notation $H^k(\Omega) = W^{k,2}(\Omega)$
  - the "2" indicates that the weak derivatives are in $L^2(\Omega)$
  - general case: $W^{k,p}(\Omega)$ has weak derivatives in $L^p(\Omega)$ up to order k

Outline

1. the Poisson equation: $-\nabla^2 u = f$
2. the Poisson equation in weak form: $\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega fv \, dx$
3. a quick sketch of the finite element method
4. defining the Sobolev spaces: $H^k(\Omega)$
5. more about $H^1(\Omega)$
6. lecture content in week 12

H^1 on an interval

- if $\Omega = (a,b)$ is an interval of the real line, then we have a simple definition using ordinary derivatives

Definition

$$H^1(a,b) = \left\{ f \in L^2(a,b) \,\middle|\, \begin{array}{c}\text{the weak derivative } f' \text{ exists}\\ \text{and } f' \in L^2(a,b)\end{array} \right\}$$

- the norm is: $\quad (\|f\|_{H^1})^2 = \int_a^b |f(x)|^2\,dx + \int_a^b |f'(x)|^2\,dx$
- but recall:
  - $L^2(a,b) \subset L^1(a,b)$, since $m((a,b)) < \infty$, and
  - $f' \in L^1(a,b) \quad \implies \quad f \in AC(a,b) \quad \text{and} \quad f(x) = f(a) + \int_a^x f'(t)\,dt$
- so, most importantly:

$$H^1(a,b) \subset C^0([a,b])$$

  - in fact: $\quad C^1([a,b]) \subsetneq H^1(a,b) \subsetneq C^0([a,b]) \subsetneq L^2(a,b)$

H^1 on an interval, via Fourier series

recall Fourier series in $L^2(-\pi,\pi)$

- using the ON sequence $\frac{1}{\sqrt{2\pi}} e^{inx}$, $n \in \mathbb{Z}$, we have:

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n e^{inx} \quad \text{with} \quad c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx,$$

- convergence is in the $\|\cdot\|_2$ norm
- we have Parseval's equality: $\|f\|_{L^2}^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$

- now consider the derivative formula: $f'(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} inc_n e^{inx}$
- ...and this norm of the derivative (*use orthonormality*):

$$\int_{-\pi}^{\pi} |f'(x)|^2\, dx = \sum_{k,n} \frac{1}{2\pi} \int_{-\pi}^{\pi} (inc_n\, e^{inx})(\overline{ikc_k\, e^{ikx}})\, dx = \sum_{k,n} nkc_n c_k \delta_{kn}$$

$$= \sum_{n=-\infty}^{\infty} n^2 |c_n|^2$$

H^1 on an interval, via Fourier series

- on Assignment 7 you show that if $f(-\pi) = f(\pi)$ and $f \in H^1(-\pi,\pi)$ then

$$f'(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} inc_n e^{inx}$$

- also, one can show that if $f \in L^2(-\pi,\pi)$ and $\sum_{n=-\infty}^{\infty} n^2|c_n|^2 < \infty$ then $f(-\pi) = f(\pi)$
- the summability condition implies a continuous periodic extension to $\mathbb{R}$

picture:

Definition (H^1 via Fourier series)

$$H^1_{per}(-\pi,\pi) = \left\{ f \in L^2(-\pi,\pi) \,\middle|\, \sum_{n=-\infty}^{\infty} n^2|c_n|^2 < \infty \right\}$$

H^k on an interval, via Fourier series

- the same logic applies for higher derivatives, for example:

$$\int_{-\pi}^{\pi} |f''|^2\,dx = \sum_{k,n} \frac{1}{2\pi} \int_{-\pi}^{\pi} ((in)^2 c_n\, e^{inx})(\overline{(ik)^2 c_k\, e^{ikx}})\,dx = \sum_{n\in\mathbb{Z}} n^4 |c_n|^2$$

Definition (H^k via Fourier series)

$$H^k_{per}(-\pi,\pi) = \left\{ f \in L^2(-\pi,\pi) \,\middle|\, \sum_{n\in\mathbb{Z}} n^{2k}|c_n|^2 < \infty \right\}$$

- there is a norm formula, by Parseval (P20 on Assignment 7):

$$\begin{aligned} \|f\|^2_{H^k} &= \int_{-\pi}^{\pi} |f|^2\,dx + \dots + \int_{-\pi}^{\pi} |f^{(k)}|^2\,dx \\ &= \sum_{n\in\mathbb{Z}} (1 + n^2 + \dots + n^{2k})|c_n|^2 = \sum_{n\in\mathbb{Z}} \left(\sum_{j=0}^{k} n^{2j} \right) |c_n|^2 \end{aligned}$$

- $H^k_{per}(-\pi,\pi)$ is often written $H^k(S^1)$

H^k on an interval, via Fourier series

Definition (H^k via Fourier series)

$$H^k(S^1) = \left\{ f \in L^2(-\pi,\pi) \,\Big|\, \sum_{n\in\mathbb{Z}} n^{2k}|c_n|^2 < \infty \right\}$$

- here you do not need weak derivatives to define Sobolev spaces
- k does not need to be an integer, but we will not pursue this
- you can extend this definition to tori, but we will not pursue this
- one needs to prove that the definition is the same (below)
- remember that these proofs use weak derivatives

Lemma

$$H^k(S^1) = \left\{ f \in H^k(-\pi,\pi) \,\Big|\, f^{(\ell)}(-\pi) = f^{(\ell)}(\pi) \text{ for } 0 \le \ell < k \right\}$$

and for $f \in H^k(S^1)$ *we have* $\|f\|_{H^k}^2 = \sum_{n\in\mathbb{Z}} \left(\sum_{j=0}^{k} n^{2j} \right) |c_n|^2$

- we have $H^1(a, b) \subset C^0([a, b])$, for a one-dimensional interval

WARNING

this inclusion is dimension-dependent

- the examples below use the unit disc D and the radial distance $r = |x|$:

$$D = \{x \in \mathbb{R}^2 \,:\, |x|^2 < 1\}$$

- the next two examples are expected, but the one after that is not (?)

are functions in $H^1(\Omega)$ continuous for $\Omega \subset \mathbb{R}^2$?

a continuous function in $H^1(D)$ … *expected*

$f(x) = \sqrt{r}$: $f \in C^0(D) \setminus C^1(D)$, and with polar coordinates $dm = r\,dr\,d\theta$,

$$\|f\|_{H^1}^2 = \int_D |f|^2 + |\nabla f|^2\,dm = 2\pi \int_0^1 \left[(\sqrt{r})^2 + \left(\frac{-1}{2\sqrt{r}}\right)^2\right] r\,dr$$
$$= 2\pi \int_0^1 (r^2 + \frac{1}{4})\,dr = \frac{7\pi}{6} < \infty$$

a discontinuous function not in $H^1(D)$ … *also expected*

$g(x) = \ln(r)$: $g \notin C^0(D)$, and

$$\|g\|_{H^1}^2 = 2\pi \int_0^1 \left[(\ln(r))^2 + \left(\frac{1}{r}\right)^2\right] r\,dr = (\text{finite}) + 2\pi \int_0^1 \frac{1}{r}\,dr = +\infty$$

pictures of $f(r), g(r)$:

are functions in $H^1(\Omega)$ continuous for $\Omega \subset \mathbb{R}^2$? NO

- define

$$p(x) = \begin{cases} \ln(|\ln(r)|), & 0 < r < 1/2 \\ \ln(\ln(2)), & 1/2 \le r < 1 \end{cases}$$

a discontinuous function which is in $H^1(D)$ ... *unexpected!*

$p(x)$: $p \notin C^0(D)$, and

$$\|p\|_{H^1}^2 = (\text{finite}) + 2\pi \int_0^{1/2} \left(\frac{-1}{r|\ln(r)|}\right)^2 r\,dr = (\text{finite}) + 2\pi \int_0^{1/2} \frac{1}{r(\ln(r))^2}\,dr$$

$$\overset{u=\ln(r)}{=} (\text{finite}) + 2\pi \int_{-\infty}^{-\ln(2)} \frac{du}{u^2} = (\text{finite}) + (\text{finite}) < \infty$$

- $p \in H^1(D)$
  - computing L^2 and H^1 norms for g and p is on Assignment 7
- generally, $H^1(\Omega) \not\subset C^0(\Omega)$ when $\Omega \subset \mathbb{R}^2$
  - but $H^2(\Omega) \subset C^0(\Omega)$ does hold
- generally, *Sobolev's inequalities* (and related) show which $H^k(\Omega)$, $\Omega \subset \mathbb{R}^d$, will contain continuous representatives

$H_0^1(\Omega)$ is the closed subspace with zero boundary values

- for any open domain $\Omega \subset \mathbb{R}^d$, the inclusion $C_c^1(\Omega) \subset H^1(\Omega)$ is clear

Definition

$$H_0^1(\Omega) = \overline{C_c^1(\Omega)}, \quad \text{where closure is in } H^1 \text{ norm}$$

- the 1D case already shows that $H_0^1(\Omega)$ is a proper closed subspace: $f(x) = 1 \in H^1(a,b)$ but $f \notin H_0^1(a,b)$ because of the FTC
  - $H_0^1(\Omega)$ is a proper closed subspace in any dimension
- picture:

Poincaré-Friedrichs inequality

- when you have zero boundary values, the only way for the function to get large is for the gradient to also be large

Theorem (Poincaré-Friedrichs inequality)

Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with Lipschitz boundary, or an open set which is bounded in one direction. There is $C > 0$, depending on Ω, so that

$$\|f\|_{L^2} \le C \|\nabla f\|_{L^2} \quad \text{for all } f \in H_0^1(\Omega).$$

picture:

a simpler norm formula on $H_0^1(\Omega)$

Theorem (Poincaré-Friedrichs inequality)

Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with Lipschitz boundary, or an open set which is bounded in one direction. There is $C > 0$, depending on Ω, so that

$$\|f\|_{L^2} \le C \|\nabla f\|_{L^2} \quad \text{for all } f \in H_0^1(\Omega).$$

- the main consequence of this inequality is that the simpler formula

$$|f|_{H^1} = \|\nabla f\|_{L^2} = \left(\int_\Omega |\nabla f|^2 \right)^{1/2}$$

  is equivalent to the full norm $\|f\|_{H^1} = \left(\int_\Omega |f|^2 + \int_\Omega |\nabla f|^2 \right)^{1/2}$ on $H_0^1(\Omega)$
- it is *not* equivalent on the whole space $H^1(\Omega)$
- generally, $|f|_{H^1}$ is called the H^1 *seminorm*
  - ...understanding that it is actually a norm on $H_0^1(\Omega)$

Poisson equation Dirichlet problems: weak forms

- now that $H_0^1(\Omega)$ is defined, we can make sense of these problems

weak formulation of Dirichlet problems

Let $\Omega \subset \mathbb{R}^d$ be an open, bounded domain with Lipschitz boundary, and suppose $f \in L^2(\Omega)$ is given. Solve:

1. homogeneous Dirichlet: Find $u \in H_0^1(\Omega)$ so that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega fv \, dx$$

   for all test functions $v \in H_0^1(\Omega)$.

2. general Dirichlet: Let $u_D \in H^1(\Omega) \cap C(\bar{\Omega})$ be any function so that $u_D = g_D$ along $\partial\Omega$. Find $\tilde{u} \in H_0^1(\Omega)$ so that

$$\int_\Omega \nabla \tilde{u} \cdot \nabla v \, dx = \int_\Omega fv \, dx - \int_\Omega \nabla u_D \cdot \nabla v \, dx$$

   for all test functions $v \in H_0^1(\Omega)$. The solution is $u = \tilde{u} + u_D \in H^1(\Omega)$.

Outline

1. the Poisson equation: $-\nabla^2 u = f$
2. the Poisson equation in weak form: $\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx$
3. a quick sketch of the finite element method
4. defining the Sobolev spaces: $H^k(\Omega)$
5. more about $H^1(\Omega)$
6. lecture content in week 12

lecture content in week 12

- most of this material is only from these slides

definitions to know:

- Laplacian
- strong form of Poisson problems (Dirichlet and Neumann)
- weak form of Poisson problems (Dirichlet and Neumann)
- continuous P_1 finite element space over a mesh of triangles
- Sobolev spaces $H^k(\Omega)$
- examples showing $H^1(\Omega) \not\subseteq C^0(\Omega)$ for $\Omega \subset \mathbb{R}^2$
- $H_0^1(\Omega)$

propositions to know:

- lemma: characterize $H^k(S^1)$ using Fourier series coefficients
- theorem: Poincaré-Friedrichs inequality for $H_0^1(\Omega)$

- Assignment 7 is posted at `bueler.github.io/fa`