

Fundamental theorems of calculus, integration-by-parts, and weak derivatives

calculations for weeks 10 & 11

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UAF Math 617 Functional Analysis

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Outline

- 1 the fundamental theorem of calculus in \mathbb{R}^1
- 2 divergence theorem on \mathbb{R}^d
- 3 integration by parts (in \mathbb{R}^1 and in \mathbb{R}^d)
- 4 weak derivatives
- 5 lecture content in weeks 10 & 11

indefinite integral of an L^1 function

- for $a < b$ real numbers, recall that

$$L^1([a, b]) = \left\{ f : [a, b] \rightarrow \mathbb{R} \mid f \text{ is measurable and } \int_a^b |f(x)| dx < \infty \right\}$$

- remember this is actually a *lie*; the elements of L^1 are *equivalence classes* up to almost everywhere
- what happens when you integrate $f \in L^1$ to get a new function?

$$F(x) = \int_a^x f(t) dt$$

- this is an *indefinite integral*
- does the fundamental theorem of calculus (FTC) apply?

$$\frac{d}{dx} \left(\int_a^x f(t) dt \right) \stackrel{?}{=} f(x)$$

- is $F'(x) = f(x)$?

fundamental theorem of calculus (FTC)

Theorem (easy direction)

If $f \in L^1([a, b])$ then $F(x) = \int_a^x f(t) dt$ is continuous, and it is differentiable almost everywhere, and $F' = f$ almost everywhere.

picture of proof:

fundamental theorem of calculus (FTC)

Theorem (easy direction)

If $f \in L^1([a, b])$ then $F(x) = \int_a^x f(t) dt$ is continuous, and it is differentiable almost everywhere, and $F' = f$ almost everywhere.

Proof when we also assume f is continuous.

If $f \in C([a, b])$ then bounded: $|f| \leq M$. Let $\epsilon > 0$. If $|x - y| < \delta = \frac{\epsilon}{M}$, and $x < y$, then

$$|F(x) - F(y)| = \left| \int_x^y f(t) dt \right| \leq \int_x^y |f(t)| dt \leq M|x - y| < M\delta = \epsilon,$$

so F is continuous. Since f is continuous at x then there is $\gamma = \gamma(x) > 0$ so that $|f(t) - f(x)| < \epsilon$ if $|t - x| < \gamma$. Choose any $0 < h < \gamma$. Then

$$\begin{aligned} \left| \frac{F(x+h) - F(x)}{h} - f(x) \right| &= \left| \frac{1}{h} \int_x^{x+h} f(t) dt - \frac{1}{h} \int_x^{x+h} f(x) dt \right| \\ &\leq \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| dt < \frac{1}{h} \int_x^{x+h} \epsilon dt = \epsilon \end{aligned}$$

(A similar argument handles $-\gamma < h < 0$.) This shows $f(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h}$; in particular the two-sided limit exists at every x . □

how the integral is well-behaved when $f \in L^1$

Lemma (“absolute continuity of the Lebesgue integral”)

Suppose $f \in L^1([a, b])$ and $\epsilon > 0$. There exists $\delta > 0$ so that if $m(E) < \delta$ then

$$\int_E |f| dm < \epsilon.$$

Proof.

Let $g_k = \min\{|f|, k\}$. This defines a sequence $0 \leq g_k \leq |f|$ so that $g_k \nearrow |f|$, but where $0 \leq g_k \leq k$ so g_k is bounded. By the Monotone Convergence Theorem,

$$\int_{[a,b]} |f| dm = \lim_{k \rightarrow \infty} \int_{[a,b]} g_k dm$$

Given $\epsilon > 0$, choose K large enough that $\int_{[a,b]} (|f| - g_K) dm < \epsilon/2$. Then for any measurable $E \subset [a, b]$:

$$\int_E |f| dm = \int_E g_K dm + \int_E (|f| - g_K) dm \leq \int_E K dm + \int_{[a,b]} (|f| - g_K) dm \leq K \cdot m(E) + \frac{\epsilon}{2}$$

Now set $\delta = \epsilon/(2K)$. If $m(E) < \delta$ then $\int_E |f| dm < \epsilon$. □

picture of the absolute continuity of the Lebesgue integral

picture:

Definition

a continuous function $f : [a, b] \rightarrow \mathbb{R}$ is *absolutely continuous* if for all $\epsilon > 0$ there is $\delta > 0$ so that if we have a finite collection of disjoint intervals $[\alpha_i, \beta_i]$, with total length less than δ :

$$\sum_{i=1}^n (\beta_i - \alpha_i) < \delta,$$

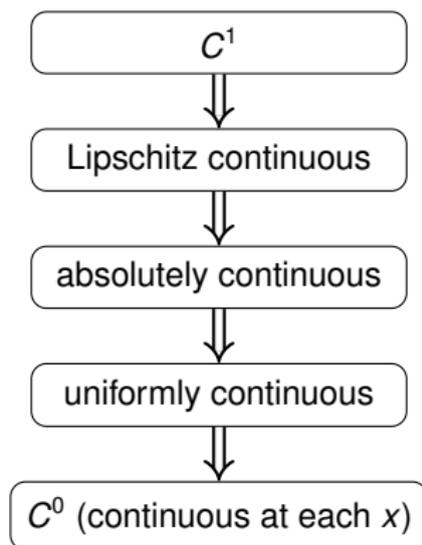
then the variation of f on those intervals is less than ϵ :

$$\sum_{i=1}^n |f(\alpha_i) - f(\beta_i)| < \epsilon$$

picture of absolute continuity

picture:

flavors of continuity for $f : [a, b] \rightarrow \mathbb{R}$



Definition

a continuous function $f : [a, b] \rightarrow \mathbb{R}$ is *Lipschitz continuous* if there is $L \geq 0$ so that

$$|f(x) - f(y)| \leq L|x - y|$$

for all $x, y \in [a, b]$

the Fundamental Theorem of Calculus again

Theorem (FTC 1)

If $f \in L^1([a, b])$ then $F(x) = \int_a^x f(t) dt$ is *absolutely continuous*, and it is differentiable almost everywhere, and $F' = f$ almost everywhere.

sketch of the proof.

Approximate f in L^1 by a continuous function g . Show that the difference $f - g$, which can be large, can't be large on a big set. (Vitali covering lemma.) Conclude. \square

Theorem (FTC 2)

If $F(x)$ is absolutely continuous then $f = F'$ exists a.e. and $f \in L^1$ and $F(x) - F(a) = \int_a^x f(t) dt$ for all x .

- take-home message: if $f \in L^1$ then its indefinite integral $F(x) = \int_a^x f(t) dt$ is a nice “very” continuous function which is differentiable a.e.
- and you can recover f from F by differentiating, at least a.e.

not the FTC, part 1: continuity is necessary ...

question: if $F : [a, b] \rightarrow \mathbb{R}$ is differentiable almost everywhere ($f = F'$ a.e.), and if $f \in L^1$, then is it true that

$$F(x) - F(a) = \int_a^x f(t) dt \quad \text{for all } x \in [a, b]?$$

answer: no

picture:

-
- my pair of formulas with $F'(x) = G'(x)$ a.e.:

$$F(x) = \begin{cases} x, & 0 < x < 1 \\ 2, & 1 < x < 2 \end{cases} \quad G(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 < x < 2 \end{cases}$$

$$F'(x) = G'(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$$

not the FTC, part 2: but not sufficient (the devil's staircase)

question: if $F : [a, b] \rightarrow \mathbb{R}$ is continuous, and if $f = F'$ exists almost everywhere, and if $f \in L^1$, then is it true that

$$F(x) - F(a) = \int_a^x f(t) dt \quad \text{for all } x \in [a, b]?$$

answer: also **no!**

picture:

Fundamental Theorem of Calculus

- 1 If $f \in L^1([a, b])$ then $F(x) = \int_a^x f(t) dt$ is in $AC([a, b])$,^a and it is differentiable a.e., and

$$\frac{d}{dx} \left(\int_a^x f(t) dt \right) = f(x)$$

for almost every x .

- 2 If $F \in AC([a, b])$ then $f = F'$ exists a.e., and $f \in L^1([a, b])$, and

$$F(x) - F(a) = \int_a^x f(t) dt$$

for all x .

^aWe define $AC([a, b]) =$ (absolutely continuous functions on $[a, b]$).

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Theorem (divergence theorem . . . as a Green's theorem)

Suppose $\Omega \subset \mathbb{R}^2$ is bounded and open. Suppose that the curve $C = \partial\Omega$ is continuously-differentiable, and orient it counter-clockwise. If $u(x, y)$ and $v(x, y)$ are continuously-differentiable functions on $\overline{\Omega}$ then

$$\iint_{\Omega} u_x + v_y \, dx \, dy = \oint_C u \, dy - v \, dx.$$

- the right side is a *curve (line) integral*; we must parameterize C as $(x(t), y(t))$:

$$\oint_C u \, dy - v \, dx = \int_0^T u(x(t), y(t))y'(t) - v(x(t), y(t))x'(t) \, dt$$

- pictures (with disconnected and not-simply-connected possibilities):

Theorem (divergence theorem)

Suppose $\Omega \subset \mathbb{R}^2$ is bounded and open, and $\partial\Omega$ is continuously-differentiable. Let \mathbf{n} be the outward normal along $\partial\Omega$. If \mathbf{V} is a continuously-differentiable vector field on $\overline{\Omega}$ then

$$\int_{\Omega} \nabla \cdot \mathbf{V} \, dm = \int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} \, ds$$

- $\mathbf{V} = \mathbf{V}(x, y)$ has components $\mathbf{V} = \langle u, v \rangle$
- $\nabla \cdot \mathbf{V} = u_x + v_y$ is the *divergence* of the vector field
- I will just use “ \int ” instead of \iint , \oint , etc.
- dm is Lebesgue measure in \mathbb{R}^2
- ds is (induced submanifold) Lebesgue measure along the 1D curve $\partial\Omega$
 - nontrivially defined! use parameterized boundary, Jacobian determinant, ...

divergence theorem example

Example

Let $\Omega = \{(x, y) : x^2 + y^2 < 1\}$ be the unit disk. Then $\partial\Omega$ is the unit circle. Let $\mathbf{V} = \langle x, y + 1 \rangle$. Is it true?

$$\int_{\Omega} \nabla \cdot \mathbf{V} \, dm \stackrel{?}{=} \int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} \, ds$$

solution steps:

- parameterize $\partial\Omega$: $(x(t), y(t)) = (\cos(t), \sin(t))$, $0 \leq t \leq 2\pi$
- $\mathbf{n} = \langle x, y \rangle = \langle \cos(t), \sin(t) \rangle$
- $ds = dt$ (arclength parameterized)
- $\mathbf{V} \cdot \mathbf{n} = \langle x, y + 1 \rangle \cdot \langle x, y \rangle = x^2 + y^2 + y = 1 + \sin(t)$
- $\nabla \cdot \mathbf{V} = 1 + 1 = 2$
- thus (*no surprise!*) the two sides are equal:

$$\int_{\Omega} \nabla \cdot \mathbf{V} \, dm = \int_{\Omega} 2 \, dm = 2\pi$$

$$\int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} \, ds = \int_0^{2\pi} 1 + \sin(t) \, dt = 2\pi$$

divergence theorem example

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divergence theorem in \mathbb{R}^d

- there is nothing intrinsically two-dimensional about the theorem!

Theorem (divergence theorem)

Suppose $\Omega \subset \mathbb{R}^d$ is bounded and open, and $\partial\Omega$ is continuously-differentiable or Lipschitz. Let \mathbf{n} be the outward normal along $\partial\Omega$. If \mathbf{V} is a continuously-differentiable vector field on $\overline{\Omega}$ then

$$\int_{\Omega} \nabla \cdot \mathbf{V} \, dm = \int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} \, ds$$

- dm is Lebesgue measure in \mathbb{R}^d
- ds is $(d - 1)$ -dim'l Lebesgue measure (submanifold measure)
- a point $x \in \mathbb{R}^d$ has coordinates x_i : $x = (x_1, \dots, x_d)$
- a vector field $\mathbf{V} : \Omega \rightarrow \mathbb{R}^d$ has components, $\mathbf{V} = \langle v_1, \dots, v_d \rangle$, and the components are scalar functions, $v_k = v_k(x_1, \dots, x_d)$
- the *divergence* of the vector field is

$$\nabla \cdot \mathbf{V} = \frac{\partial v_1}{\partial x_1} + \dots + \frac{\partial v_d}{\partial x_d} = \sum_{k=1}^d \frac{\partial v_k}{\partial x_k}$$

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picture of divergence theorem in \mathbb{R}^3 (and hints of proof)

picture in \mathbb{R}^3 :

most-famous usage in physics

- Gauss' law for electric fields might look like this in a physics text.

Gauss' law ("integral form")

Suppose $\mathbf{E} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is the (continuously-differentiable) electric field. Suppose $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$. Then

$$\oiint_{\partial\Omega} \mathbf{E} \cdot \mathbf{n} \, ds = \frac{Q}{\epsilon_0}$$

where Q is the net enclosed charge within Ω , and $\epsilon_0 > 0$ is a constant.

- apply the divergence theorem on the left, and write Q as the integral of a charge density function $\rho : \mathbb{R}^3 \rightarrow \mathbb{R}$, $\rho \in L^1$:

$$\int_{\Omega} \nabla \cdot \mathbf{E} \, dm = \frac{1}{\epsilon_0} \int_{\Omega} \rho \, dm$$

- since this applies over any $\Omega \subset \mathbb{R}^3$, we get Gauss' law in "PDE form":

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

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Gauss' law (“integral form”)

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fundamental theorems in any dimension

- the Fundamental Theorem of Calculus, in any dimension, says that the integral of a derivative can be computed by integrating over the boundary:

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega \quad (\text{Stokes theorem})$$

- making this precise requires defining *chains* Ω and *differential forms* ω
- which I'm not going to do
- a stack of Fundamental Theorems for $\Omega \subset \mathbb{R}^d$, in various dimensions:

$$\int_a^b F' dm = F(b) - F(a) \quad (\text{interval domain, two-point boundary})$$

$$\iint_{\Omega} \nabla \cdot \mathbf{V} dm = \oint_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} ds \quad (\text{planar domain, closed-curve boundary})$$

$$\iiint_{\Omega} \nabla \cdot \mathbf{V} dm = \oiint_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} ds \quad (\text{solid domain, closed-surface boundary})$$

$$\int_{\Omega} \nabla \cdot \mathbf{V} dm = \int_{\partial\Omega} \mathbf{V} \cdot \mathbf{n} ds \quad (\mathbb{R}^d \text{ domain, } (d-1)\text{-dim'l closed boundary})$$

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product rule and integration-by-parts in \mathbb{R}^1

- this is very familiar!
- suppose $f(x), g(x)$ are continuously-differentiable (or absolutely continuous)
- then:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

- if we integrate both sides from a to b , and apply FTC on left:

$$\left[f(x)g(x) \right]_a^b = \int_a^b f'(x)g(x) dx + \int_a^b f(x)g'(x) dx$$

- rearranging, we get integration-by-parts:¹

$$\int_a^b f(x)g'(x) dx = \left[f(x)g(x) \right]_a^b - \int_a^b f'(x)g(x) dx$$

- it suffices for f, g to be absolutely continuous (and thus $f', g' \in L^1$)

¹also known as $\int u dv = uv - \int v du$, but then you put in limits of integration ...

product rules in \mathbb{R}^d

- notation:
 - $x \in \mathbb{R}^d$ has coordinates: $x = (x_1, \dots, x_d)$
 - $f, g : \mathbb{R}^d \rightarrow \mathbb{R}$ are scalar functions
 - \mathbf{V} has components: $\mathbf{V} = \langle v_1, \dots, v_d \rangle$ where v_k are scalar functions
 - gradient $\nabla f = \left\langle \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_d} \right\rangle$ and divergence $\nabla \cdot \mathbf{V} = \sum_{k=1}^d \frac{\partial v_k}{\partial x_k}$
- there are **2 priority product rules** in \mathbb{R}^d to know:²

product rule 1: partial derivatives (of scalar functions)

$$\frac{\partial}{\partial x_k} (fg) = \frac{\partial f}{\partial x_k} g + f \frac{\partial g}{\partial x_k}$$

product rule 2: divergence/gradient

$$\nabla \cdot (f\mathbf{V}) = \nabla f \cdot \mathbf{V} + f \nabla \cdot \mathbf{V}$$

²there are also product rules for the curl (" $\nabla \times \mathbf{V}$ "), and/or for the exterior derivative and differential-forms wedge product (" d " and " $\omega \wedge \nu$ "); know these for electricity and magnetism!

integration-by-parts in \mathbb{R}^d

- the **2 priority integration-by-parts rules** in \mathbb{R}^d come from the two product rules . . . via the same logic as in \mathbb{R}^1 !
- the first rule is simplest and most useful *when the boundary integral is zero*, thus the assumption of compact support
- because of the assumption of compact support, the boundary of Ω is allowed to be weird

integration-by-parts 1: partial derivatives

if $\Omega \subset \mathbb{R}^d$ is open, and if $f, g \in C_c^1(\Omega)$ then

$$\int_{\Omega} \frac{\partial f}{\partial x_k} g \, dm = - \int_{\Omega} f \frac{\partial g}{\partial x_k} \, dm$$

- but what is “ $C_c^1(\Omega)$ ”?

Definition

- If $\Omega \subset \mathbb{R}^d$ is open, we say that a function $f : \Omega \rightarrow \mathbb{R}$ *has compact support* if

$$\text{supp}(f) = \overline{\{x \in \Omega : f(x) \neq 0\}}$$

is a compact subset of Ω .

- $C_c(\Omega) = \{\mathbb{R}\text{-valued continuous functions which have compact support}\}$
- $C_c^n(\Omega) = \{f \in C_c(\Omega) : \text{partial derivatives up to order } n \text{ are in } C_c(\Omega)\}$
- $C_c^\infty(\Omega) = \{f \in C_c(\Omega) : \text{partial derivatives of all orders are in } C_c(\Omega)\}$

- note $C_c^0(\Omega) = C_c(\Omega)$
- $C_c^\infty(\Omega)$ functions are the nicest! you can differentiate and/or integrate over Ω as much as you want

- the second formula, easily derived from the divergence theorem, is sometimes called a *Green's identity* or similar
- it requires the boundary to have a normal direction, so the boundary cannot be too weird

integration-by-parts 2: divergence/gradient

if $\Omega \subset \mathbb{R}^d$ is open, bounded, and has continuously-differentiable or Lipschitz boundary, and if f and the components of \mathbf{V} are in $C^1(\overline{\Omega})$, then

$$\int_{\Omega} f \nabla \cdot \mathbf{V} \, dm = \int_{\partial\Omega} f \mathbf{V} \cdot \mathbf{n} \, ds - \int_{\Omega} \nabla f \cdot \mathbf{V} \, dm$$

compact support and integration-by-parts

pictures:

left: for partial derivatives

right: for divergence/gradient

application of integration-by-parts: weak form of a PDE

- suppose we want to solve the *Poisson equation*, a partial differential equation (PDE), on a domain $\Omega \subset \mathbb{R}^d$, with given data $f \in L^1(\Omega)$:

$$-\nabla^2 u \stackrel{*}{=} f$$

- we seek $u : \Omega \rightarrow \mathbb{R}$, which also satisfies some boundary conditions (below)
- notation: $\nabla^2 u = \nabla \cdot (\nabla u)$ is the *Laplacian* of u ; it is also a scalar function
- concept: $*$ is true if and only if arbitrary integrals of the sides are equal
- multiply both sides of $*$ by another function v , a *test function*, expanding the Laplacian, and integrate over Ω :

$$-\int_{\Omega} \nabla \cdot (\nabla u) v \, dm = \int_{\Omega} f v \, dm$$

- apply integration-by-parts 2 for divergence/gradient:

$$\int_{\Omega} \nabla u \cdot \nabla v \, dm - \int_{\partial\Omega} v \nabla u \cdot \mathbf{n} \, ds = \int_{\Omega} f v \, dm$$

application of integration-by-parts: weak form of a PDE

- from last slide, Poisson's equation $-\nabla^2 u = f$ has become

$$\int_{\Omega} \nabla u \cdot \nabla v \, dm - \int_{\partial\Omega} v \nabla u \cdot \mathbf{n} \, ds = \int_{\Omega} f v \, dm$$

- the boundary term currently involves the solution u
- however, it becomes data if either
 - we set the normal derivative, $\nabla u \cdot \mathbf{n} = g_N$ (*Neumann condition*), or
 - we set $u = g_D$ and $v = 0$ along the boundary (*Dirichlet condition*)
- if we choose homogeneous Dirichlet conditions ($v = 0$ and $u = 0$ along boundary) then, essentially, we have equivalent forms of Poisson's equation:

$$-\nabla^2 u = f$$

strong form



$$\int_{\Omega} \nabla u \cdot \nabla v \, dm = \int_{\Omega} f v \, dm$$

weak form

- the weak form is what we will use for finite element approximations

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simplifying awkward partial derivative notation

- integration by parts will allow us to differentiate very general functions
- but first: multi-indices simplify notation
- let $\mathbb{Z}^+ = \{0, 1, 2, \dots\} = \{0\} \cup \mathbb{N}$
- suppose $\alpha = (\alpha_1, \dots, \alpha_d)$ is a **multi-index**:

$$\alpha \in \mathbb{Z}^+ \times \dots \times \mathbb{Z}^+ = (\mathbb{Z}^+)^d,$$

- $|\alpha| = \alpha_1 + \dots + \alpha_d$ is the *order* of α

Definition

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$$

- for example on $f : \mathbb{R}^3 \rightarrow \mathbb{R}$:

$$\alpha = (3, 0, 0) \quad \rightarrow \quad D^\alpha f = \frac{\partial^3 f}{\partial x_1^3} \quad (\text{3rd-order})$$

$$\alpha = (0, 1, 1) \quad \rightarrow \quad D^\alpha f = \frac{\partial^2 f}{\partial x_2 \partial x_3} \quad (\text{2nd-order mixed})$$

key idea about $C_c^\infty(\Omega)$

- suppose $\Omega \subset \mathbb{R}^d$ is open
- for any function $\varphi : \Omega \rightarrow \mathbb{R}$, recall that we defined the closed set

$$\text{supp}(\varphi) = \overline{\{x \in \Omega : \varphi(x) \neq 0\}}$$

- recall that for $C_c^\infty(\Omega)$ functions we may differentiate as much as we want
- if $\varphi \in C_c^\infty(\Omega)$ then the support of φ is compact
- if $\varphi \in C_c^\infty(\Omega)$ then the support of *any* $D^\alpha \varphi$, for any α , is compact

main idea:

if $\varphi \in C_c^\infty(\Omega)$ then the integral over $K = \text{supp}(D^\alpha \varphi)$, for any α , of any continuous or integrable function, is well-defined and finite:

$$g \in L^1(\Omega) \implies \int_{\Omega} g D^\alpha \varphi \, dm = \int_K g D^\alpha \varphi \, dm \quad \text{is finite}$$

- we will use $\varphi \in C_c^\infty(\Omega)$ as a *test function* to define a derivative of a much more general kind of function . . . if the derivative exists

Definition

if $f \in L^1(\Omega)$, and if $g \in L^1(\Omega)$ satisfies

$$\int_{\Omega} g \varphi \, dm = (-1)^{|\alpha|} \int_{\Omega} f D^\alpha \varphi \, dm \quad \text{for all } \varphi \in C_c^\infty(\Omega)$$

then we say g is the α *th weak derivative* of f , and we write $D^\alpha f = g$ a.e.

- this definition uses “integration-by-parts 1” for partial derivatives
- because we don't care if functions are (globally) integrable in this definition, we may loosen $f, g \in L^1(\Omega)$ to $f, g \in L^1_{\text{loc}}(\Omega)$

- we will use $\varphi \in C_c^\infty(\Omega)$ as a *test function* to define a derivative of a much more general kind of function . . . if the derivative exists

Definition

if $f \in L^1_{\text{loc}}(\Omega)$, and if $g \in L^1_{\text{loc}}(\Omega)$ satisfies

$$\int_{\Omega} g \varphi \, dm = (-1)^{|\alpha|} \int_{\Omega} f D^\alpha \varphi \, dm \quad \text{for all } \varphi \in C_c^\infty(\Omega)$$

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weak derivatives: 3 definitions to know

Definition

if $\alpha \in (\mathbb{Z}^+)^d$ then $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$

Definition

$$L^1_{\text{loc}}(\Omega) = \left\{ f : \Omega \rightarrow \mathbb{R} \mid \begin{array}{l} f \text{ is measurable, and} \\ \int_K |f| dm < \infty \text{ whenever } K \subset \Omega \text{ is compact} \end{array} \right\}$$

Definition

if $f \in L^1_{\text{loc}}(\Omega)$, and if $g \in L^1_{\text{loc}}(\Omega)$ satisfies

$$\int_{\Omega} g \varphi dm = (-1)^{|\alpha|} \int_{\Omega} f D^\alpha \varphi dm \quad \text{for all } \varphi \in C_c^\infty(\Omega)$$

then we say g is the α th weak derivative of f , and we write $D^\alpha f = g$ a.e.

weak derivatives: Example 1

- suppose $\Omega \subset \mathbb{R}^d$ is open and $f \in C^k(\Omega)$
- then the usual partial derivative $D^\alpha f$ exists, as long as $|\alpha| \leq k$
- integrate the usual derivative $D^\alpha f$ over Λ , using integration-by-parts 1, for each partial derivative in $\alpha = (\alpha_1, \dots, \alpha_d)$, a total of $|\alpha|$ times:

$$\begin{aligned}\int_{\Omega} (D^\alpha f) \varphi \, dm &= - \int_{\Omega} D^{\tilde{\alpha}} f D^{(1,0,\dots,0)} \varphi \, dm \\ &= \dots = (-1)^{\alpha_1} \int_{\Omega} D^{\tilde{\tilde{\alpha}}} f D^{(\alpha_1,0,\dots,0)} \varphi \, dm \\ &= \dots \\ &= (-1)^{|\alpha|} \int_{\Omega} f (D^\alpha \varphi) \, dm\end{aligned}$$

- thus the weak derivative g exists, namely $g = D^\alpha f$

weak derivatives: Example 2

- let's do a small \mathbb{R}^1 example
- I will draw pictures on the right
- consider this continuous function:

$$f(x) = \begin{cases} |x|, & -1 < x < 1 \\ 1, & |x| \geq 1 \end{cases}$$

- one can show that, for the ordinary derivative $D^\alpha = \frac{d}{dx}$, $\alpha = (1)$, we have

$$\int_{\mathbb{R}} g\varphi \, dm = - \int_{\mathbb{R}} f\varphi' \, dm$$

for every $\varphi \in C_c^\infty(\mathbb{R})$, where

$$g(x) = -\mathbb{1}_{(-1,0)}(x) + \mathbb{1}_{(0,1)}(x)$$

- thus: $g = f'$, as a weak derivative, which is no surprise
- note that g is only defined up to a set of measure zero

weak derivatives: Example 3

- part of the concern with weak derivatives is local integrability
- for example, consider this $f \in L^1_{\text{loc}}(\mathbb{R}^2)$:

$$f(x) = \frac{1}{r}$$

on \mathbb{R}^2 , where $x = (x_1, x_2)$ and $r = (x_1^2 + x_2^2)^{1/2}$

- note f is continuous, except at $x = 0$
- let B_ϵ be the open ball of radius $\epsilon > 0$ centered at the origin
- $f \in L^1_{\text{loc}}(\mathbb{R}^2)$ because this integral gives a finite answer:

$$\int_{B_\epsilon} |f| \, dm = \int_{B_\epsilon} \frac{1}{r} \, dm = \int_0^{2\pi} \int_0^\epsilon \frac{1}{r} r \, dr \, d\theta = 2\pi \int_0^\epsilon 1 \, dr = 2\pi\epsilon < +\infty$$

- however, the only candidate (a.e.) for the weak derivative is

$$g(x) = (\nabla f)(x) = -\frac{\langle x_1, x_2 \rangle}{r^3},$$

and $g \notin L^1_{\text{loc}}(\mathbb{R}^2)$ because

$$\int_{B_\epsilon} |g| \, dm = \int_{B_\epsilon} \frac{1}{r^2} \, dm = 2\pi \int_0^\epsilon \frac{1}{r} \, dr = +\infty$$

weak derivatives: Example 4

- however, weak derivatives can exist for functions with singularities!
- for example, consider a new $f \in L^1_{\text{loc}}(\mathbb{R}^2)$, which again is continuous, except at $x = 0$:

$$f(x) = \ln(r)$$

- $f \in L^1_{\text{loc}}(\mathbb{R}^2)$ because this integral gives a finite answer:

$$\int_{B_\epsilon} |f| \, dm = \int_{B_\epsilon} |\ln(r)| \, dm = -2\pi \int_0^\epsilon \ln(r) r \, dr = \frac{\pi\epsilon^2}{2} (1 - 2\ln(\epsilon)) < +\infty$$

- now the only candidate for the weak derivative is $g(x) = (\nabla f)(x)$, namely

$$g(x) = -\frac{\langle x_1, x_2 \rangle}{r^2},$$

and $g \in L^1_{\text{loc}}(\mathbb{R}^2)$ because

$$\int_{B_\epsilon} |g| \, dm = \int_{B_\epsilon} \frac{1}{r} \, dm = 2\pi \int_0^\epsilon 1 \, dr = 2\pi\epsilon < +\infty$$

Outline

1. the fundamental theorem of calculus in \mathbb{R}^1
2. divergence theorem on \mathbb{R}^d
3. integration by parts (in \mathbb{R}^1 and in \mathbb{R}^d)
4. weak derivatives
5. lecture content in weeks 10 & 11

lecture content in weeks 10 & 11

- part of this material is only from these slides

to know from slides:

- Lebesgue's FTC on $[a, b]$, with $f \in L^1$ and $F \in AC$
 - divergence theorem in \mathbb{R}^d
 - integration-by-parts in \mathbb{R}^d
 - multi-index notation for derivatives in \mathbb{R}^d , and definition of $C_c^\infty(\Omega)$
 - weak derivative in \mathbb{R}^d
- ... but part will be from Chapter 5 in Saxe, sections 5.1 and 5.2

to define:

- linear operator
- bounded linear operator
- operator norm
- dual space (*I will add this closely-related topic*)

to prove:

- a linear operator is continuous if and only if it is bounded (Theorem 5.2)
 - the bounded linear operators $\mathcal{B}(X, Y)$ form a normed vector space under operator norm (Theorem 5.3)
 - if Y is complete then $\mathcal{B}(X, Y)$ is complete (Theorem 5.4)
 - the dual space is a Banach space
 - dual vectors on a Hilbert space can be represented by elements of the Hilbert space (Riesz representation theorem)
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- Assignment 6 is posted at bueler.github.io/fa