Integration is a linear functional

week 1 calculation

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UAF Math 617 Functional Analysis

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welcome to functional analysis!

motivation for **functional analysis**:

- theory behind *partial differential equations*
- theory behind *Fourier series* and *Fourier/Laplace transform*
- theory behind *quantum mechanics*
- when made finite-dimensional, within a flexible framework, functional analysis practically becomes the *finite element method*

welcome to functional analysis!

plan

each week starts with a concrete calculation, sometimes related to finite elements, then the rest of the week will cover the theory supporting that calculation

- most of the theory will be from the textbook, K. Saxe, *Beginning Functional Analysis*
- when in doubt, assume that *material on the Exams* is from the theory and the textbook, not the more numerical/calculational parts
    - do not expect to become an expert at finite elements just from this course
- this plan is an *experiment* . . . I have taught functional analysis before but not this way

Outline

Definition (continuity)

- a function $f : [a,b] \to \mathbb{R}$ is *continuous at* $x \in [a,b]$ if for all $\epsilon > 0$ there is $\delta > 0$ so that

$$|y - x| < \delta \text{ and } y \in [a,b] \quad \implies \quad |f(y) - f(x)| < \epsilon$$

- equivalently: $\quad f(x) = \lim_{y \to x} f(y) \quad$ (where the limit exists!)
- a function $f : [a,b] \to \mathbb{R}$ is *continuous* if it is continuous at each $x \in [a,b]$

- picture:

Definition

$$C([a,b]) = \{f : [a,b] \to \mathbb{R} \,:\, f \text{ is continuous}\}$$

- $C([a,b])$ is a set in which each element is a function f
- standard example $a = 0$, $b = 1$: $\quad C([0,1])$
- $C([0,1])$ is an infinite set because it contains distinct elements 1, x^1, x^2, x^3, ...
- picture:

$C([a,b])$ is vector space

- what is the definition of a *vector space*?
- "linear space" is a synonym for vector space

Theorem

$V = C([a,b])$ is a (real) vector space

Proof.

If $f,g \in V$ then $h(x) = f(x) + g(x) = g(x) + f(x)$ is a continuous function, so V is closed under addition. If $\alpha \in \mathbb{R}$ then $k(x) = \alpha f(x)$ is a continuous function, so V is closed under scalar multiplication. The zero function is a continuous function, and $f(x) + 0 = f(x)$, so V has an additive identity. Also $1 f(x) = f(x)$, so there is scalar identity. And distribution rules hold: $\alpha(f+g) = \alpha f + \alpha g$, $\alpha(\beta f) = (\alpha\beta) f$. □

Definition (ordinary derivative)

given $f : [a, b] \to \mathbb{R}$ and $c \in (a, b)$ define $\quad f'(c) = \lim_{h \to 0} \dfrac{f(c + h) - f(c)}{h}$
to be the *derivative of f at c*, but only *if* the limit exists

- for $f(x) = x(1 - x) \in C([0, 1])$ we have $f'(c) = 1 - 2c$ for $c \in (0, 1)$
- for $f(x) = |1 - 2x| \in C([0, 1])$, the limit to define $f'(0.5)$ does not exist

picture:

definite integral

Definition (Riemann's integral)

for meshes $a = x_0 < x_1 < \cdots < x_j < \cdots < x_n = b$, and evaluation points $x_j^* \in [x_{j-1}, x_j]$, define
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{j=1}^n f(x_j^*)\,(x_j - x_{j-1})$$
as the *Riemann integral* of f over $[a,b]$, but only *if* the limit exists

- the limit process for an integral is more subtle than the limit process for continuity or derivatives
  - this limit *does not* exist for some (non-continuous) functions
- Riemann's is not the only definition of an integral
  - we will get to Lebesgue's integral later
  - it agrees with Riemann's when they are both defined
  - we will use Lebesgue integration routinely, but only lightly cover its details

Theorem

if $f : [a,b] \to \mathbb{R}$ is continuous then $\int_a^b f(x)\,dx \in \mathbb{R}$ is well-defined

integration example

- picture:

- example: compute $\int_0^3 x \, dx$ from the definition

fundamental theorem of calculus

Theorem

if $f : [a, b] \to \mathbb{R}$ is continuous and if $F : [a, b] \to \mathbb{R}$ satisfies $F'(x) = f(x)$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

- example: redo integral on last slide

more integration examples

- example: compute $\int_0^1 x^k \, \frac{1}{1+x^2} \, dx$ for $k = 0, 1, 2$

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claim: for $k \in \mathbb{Z}^+ = \{0, 1, 2, \dots\}$, the values $c_k = \int_0^1 x^k \frac{1}{1+x^2} \, dx$ are

$$c_k = \frac{\pi}{4}, \frac{\ln(2)}{2}, 1 - \frac{\pi}{4}, \dots = 0.78539, 0.34657, 0.21460, 0.15342, 0.11873, 0.09657, \dots$$

non-integrable functions

- describe a function $f : [0, 1] \to \mathbb{R}$ for which $\int_0^1 f(x)\,dx$ does not exist:

$$\sum_{j=1}^{n} f(x_j^*)\,(x_j - x_{j-1}) =$$

(so the Riemann integral $\int_0^1 f(x)\,dx = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j^*)\,(x_j - x_{j-1})$ does not exist)

- *the moral:* For both derivatives and integrals, we need to restrict the input functions so that these limits are well-defined. But integration is more flexible: continuity is sufficient.

Outline

linear functionals on $C([0,1])$

Definition (linear functional)

a function $\ell : C([0,1]) \to \mathbb{R}$ is a *linear functional* if

$$\ell[\alpha f + \beta g] = \alpha \ell[f] + \beta \ell[g]$$

for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in C([0,1])$

- example: $\ell[f] = f(0.4)$
- example?: $\ell[f] = 4f$
- example?: $\ell[f] = 4f(0) + 2f(0.6)$
- example?: $\ell[f] = \int_{0.4}^{0.6} f(t)\,dt$
- example?: $\ell[f] = \int_{0.4}^{x} f(t)\,dt$

integration against a weight

- example: suppose $w \in C([0,1])$ is fixed. then

$$\ell[f] = \int_0^1 f(x) w(x)\, dx$$

  is a linear functional on $C([0,1])$

Proof.

□

- example?: $\ell[f] = \int_0^1 f(x)\, dx$
- example?: $\ell[f] = \int_0^1 \frac{f(t)}{1+t^2}\, dt$
- example?: $\ell[f] = \int_x^{0.6} f(t)\, dt$
- example?: $\ell[f] = \int_{0.4}^{0.6} f(t)\, dt$

example

let $w(x) = \dfrac{1}{1+x^2}$. consider the linear functional

$$\ell[f] = \int_0^1 f(x) \frac{1}{1+x^2}\, dx$$

what are its values?

$f(x) = x^k$:

$f(x) = \sin(x)$: $= 0.3217935447$, says claude

$f(x) = \sin(k\pi x)$:

- generally we have to *approximate* to answer such questions . . .

questions about linear functionals

regularity question

we don't need f and w to be continuous for this to be well-defined:

$$\ell[f] = \int_0^1 f(x)w(x)\,dx$$

how general can these functions be? can they be discontinuous? completely arbitrary? is ℓ itself continuous?

recovery question

suppose ℓ has a formula $\ell[f] = \int_0^1 f(x)w(x)\,dx$, but we don't know $w(x)$. can we recover $w(x)$ from the values $\ell[f]$ for various f?

conceptual question

consider the space V' of all linear functionals on $V = C([0,1])$. is it a vector space? how do we concretely understand this abstractly-defined space? is each $\ell \in V'$ represented by a $w(x)$?

recovery example 1

- consider:

$$\ell[f] = \int_0^1 f(x)w(x)\,dx$$

- suppose we know that $w(x)$ is a degree 4 polynomial
- how to recover $w(x)$?

recovery example 2

- consider:

$$\ell[f] = \int_0^1 f(x)w(x)\,dx$$

- suppose $w(x) = 1/(1 + x^2)$ in fact, but pretend we don't know that
- how to recover $w(x)$?

  try $f(x) = x^k$?:

  try $f(x) = \sin(k\pi x)$?:

recovery example 2

- consider:

$$\ell[f] = \int_0^1 f(x) w(x)\, dx$$

- suppose $w(x) = 1/(1 + x^2)$ in fact, but pretend we don't know that
- how to recover $w(x)$?

  try hat functions on a uniform mesh:

question: is $\ell[f] = f'(0.5)$ a linear functional on $C([0,1])$?

discuss!

Outline

numerical integration rules

- **trapezoid rule** for m equal intervals on $[0,1]$:

$$\int_0^1 f(x)\,dx \approx T_m(f) = \frac{h}{2}\Big(f(x_0) + 2f(x_1) + \dots + 2f(x_{m-1}) + f(x_m)\Big)$$

  where $h = 1/m$ and $x_j = jh$
- **Simpson's rule** for *even* m intervals, and same h, x_j:

$$\int_0^1 f(x)\,dx \approx S_m(f) = \frac{h}{3}\Big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \dots + 4f(x_{m-1}) + f(x_m)\Big)$$

- ...there are many more rules, and many are better, but they have the same flavor ...

numerical integration is a linear functional is a row vector

- suppose we list values of f in a column vector: $$f \approx \begin{bmatrix} f(x_0) \\ f(x_1) \\ \vdots \\ f(x_m) \end{bmatrix}$$
- then

$$T_m(f) =$$

$$S_m(f) =$$

finite-dimensional vector spaces

- suppose we have a real, finite-dimensional vector space V
- suppose we have a basis for V: $\mathcal{B} = \{b_1, \dots, b_m\} \subset V$
- write $f \in V$ in this basis:

$$f = f_1 b_1 + \dots f_m b_m \qquad \text{for } f_j \in \mathbb{R}$$

- the coefficients are usually written in a column vector:

$$(f)_{\mathcal{B}} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix}$$

linear functionals in finite dimensions

Theorem

if V is a real, finite-dimensional vector space with a basis $\mathcal{B} = \{b_1, \dots, b_m\}$, and if $f = f_1 b_1 + \dots f_m b_m$ in this basis, then for any linear functional $\ell : V \to \mathbb{R}$ there are values ℓ_j, $j = 1, \dots, m$, so that

$$\ell[f] = \begin{bmatrix} \ell_1 & \ell_2 & \dots & \ell_m \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{bmatrix} = \sum_{j=1}^{m} \ell_j f_j$$

- in other words, we may represent linear functions by *row vectors* when the vectors themselves are, as usual, *column vectors*
- ...and then $\ell[f]$ is computed by usual rules of matrix-matrix multiplication

Outline

theory on Wednesday and Friday

definitions:

- metric space
- norms
- normed vector space
- inner products
- sequence spaces ℓ^p, and norms $\|(x_n)\|_p$ and $\|(x_n)\|_\infty$

$=$ Chapter 1 in Saxe

Assignment 1 posted on `bueler.github.io/fa`