Assignment 8

Due Friday 24 April at the beginning of class

This Assignment is based mostly on the week 12 and week 13 slides.

DO THE FOLLOWING PROBLEMS.

P23. *Refer to Chapter 4 of the textbook*[1] *as needed.*

(a) Consider $f \in L^2(-\pi, \pi)$ expanded using the complete ON sequence $\{\frac{1}{\sqrt{2\pi}} e^{inx}\}_{n\in\mathbb{Z}}$. The coefficients c_n in the expansion

$$f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

are generally complex, but show that if f is real-valued then

$$c_{-n} = \overline{c_n}.$$

(b) Again assume that f is real-valued. Use the substitution $c_n = \frac{\sqrt{2\pi}}{2}(a_n - ib_n)$, with $a_n, b_n \in \mathbb{R}$, to show the fully-real expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\theta) + b_n \sin(n\theta).$$

Starting from the formula for c_n as an integral of f, derive the simplified, real formulas which compute a_n and b_n.

P24. (a) Suppose $\Omega \subset \mathbb{R}^d$ is open and bounded, with Lipschitz boundary so that the outward normal vector field $\mathbf{n}$ is well-defined and the divergence theorem can be applied. Suppose $\gamma > 0$ and $f \in L^2(\Omega)$. Consider the following strong form of the homogeneous Neumann problem for the Helmholtz equation:

$$-\nabla^2 u + \gamma u = f \quad \text{on } \Omega, \qquad \nabla u \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega.$$

Derive the weak form, where test functions and the solution are from $H^1(\Omega)$.

A proof of well-posedness of the homogeneous Dirichlet problem for the Poisson equation, using the Riesz-Fréchet representation theorem, was shown in the week 13 slides. Please follow this strategy. However, you will not *need the Poincaré-Friedrichs inequality.*

(b) Show that the weak form, derived in part **(a)**, is well-posed for $u \in H^1(\Omega)$.

P25. *Both parts can be answered using strong forms.*

(a) Let $\Omega = (0, \pi)^2 \subset \mathbb{R}^2$ be the open square. Show that the functions

$$\phi_{jk}(x, y) = \cos(jx)\cos(ky), \quad j, k \in \mathbb{N} \cup \{0\},$$

[1] K. Saxe, *Beginning Functional Analysis*, Springer 2010.

are eigenfunctions of the Laplacian, and find their eigenvalues. Also show that each ϕ_{jk} satisfies homogeneous Neumann conditions along $\partial\Omega$.

(b) I claim that, for $f = 0$ and for certain negative values of γ, the eigenfunctions of part **(a)** show that, on this square Ω, the Helmholtz problem in **P24** is *not* well-posed. Find the values of γ where this is true.

The Helmholtz problem in **P24***, with the restriction $\gamma > 0$, is often called* good Helmholtz.

P26. **(a)** Again let $\Omega = (0, \pi)^2 \subset \mathbb{R}^2$. Consider the functions

$$\psi_{jk} = \sin(jx)\sin(ky), \quad j, k \in \mathbb{N},$$

Show that they form a doubly-indexed orthogonal sequence in $L^2(\Omega)$, which are eigenfunctions of the Laplacian.

I claim that, with appropriate normalization, the eigenfunctions ψ_{jk} form a complete ON sequence. You do not need to prove this, but you will use it below.

(b) Additionally suppose $f \in L^2(\Omega)$. Consider the Poisson equation with homogeneous Dirichlet boundary conditions:

$$-\nabla^2 u = f \quad \text{on } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Use the eigenfunctions ψ_{jk} to construct the solution u to this problem.

(c) Show that $u \in H^1(\Omega)$.

P27. *The week 13 slides contain a theorem showing how to solve Laplace's equation on the unit disc $D \subset \mathbb{R}^2$. The Dirichlet boundary condition data is $g \in L^2(\partial D) = L^2(S^1)$.*

Suppose that $g \in L^2(S^1)$, real-valued, is written as a classical Fourier series, namely in the form of **P23 (b)**. Define

$$u(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n r^n \cos(n\theta) + b_n r^n \sin(n\theta)$$

for $0 \le r \le 1$ and $-\pi < \theta \le \pi$.

(a) Let $\epsilon > 0$. Prove that the series for u converges uniformly on

$$\bar{D}_\epsilon = \left\{x \in \mathbb{R}^2 \,:\, 0 \le r \le 1 - \epsilon, -\pi < \theta \le \pi, \text{ for } x \text{ in polar coordinates}\right\},$$

a smaller closed disc.

(b) Conclude that u is continuous on D.

(c) Prove that u has boundary values g in the sense that, in the $L^2(S^1)$ norm,

$$\lim_{r \to 1^-} u(r, \theta) = g(\theta).$$